An Authorization Logic with Explicit Time[1]

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Abstract

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1 Introduction

Most secure systems restrict operations that users, machines, and other principals can perform on files and other resources. A reference monitor authorizes (or denies) requests to access resources, in consultation with a set of rules called the security policy. Time is central to most policies. A student, for instance, may be allowed to access course related material *only* during the specific semester that she is registered for the class.

In practice, security policies are often large and complicated, necessitating formal mechanisms for both their enforcement and their analysis. Although several trust management frameworks [8, 9, 24, 27–29, 32], languages [7, 14], and access control logics [1, 2, 12, 18, 19, 25, 26] have been proposed for enforcing and sometimes for reasoning about access control policies, these proposals rarely handle time explicitly, either omitting it altogether, or leaving it to an external enforcement mechanism. As a result, policies with complex time-dependent relationships cannot be expressed, and, in other cases, reasoning accurately about time is extremely difficult.

The purpose of this report is to bridge this gap between using time in practice and reasoning about it; we propose an authorization logic that allows explicit mention of time, making it easier to reason about the time-dependent consequences of policies. This logic combines ideas from an existing authorization logic [18, 19] with ideas from both hybrid logics [11, 33] and constraint-based logics [23, 34] to allow formulas of the form $A @ I$ where A is a proposition and I is the time interval on which it holds. Following earlier proposals, we make the logic constructive to keep evidence as direct as possible. We also include linearity to model consumable authorizations; their use is illustrated in our examples. We call the logic η-logic (pronounced *eta logic* for *Explicitly Timed Authorization logic*).

η-logic is strictly more expressive than existing logics for access control, because policies with complex time-dependent relationships can be expressed in it, which is impossible in logics proposed hitherto. For instance, the following policy can be expressed in η-logic: "If an employee requests a parking space before the end of a month, she will be given a parking permit valid throughout the next month." It is difficult to imagine how such a policy could be expressed unless time is allowed explicitly in formulas. The policies described in section 4 illustrate similar complex time-based relationships.

Our principal interest in designing η-logic is its deployment with proof-carrying authorization (PCA) [3–5]. In the PCA paradigm, security policies are formalized in a logic, and each access request is accompanied by a formal proof establishing that authorization for the request follows logically from the policies. The reference monitor verifies the correctness of the proof, and allows or denies access accordingly. PCA provides a flexible mechanism for access control in distributed systems. In existing approaches the proof presented to the reference monitor establishes that the requester is allowed to access the resource in question, leaving the validity of the proof *at the time of request* to a separate enforcement check. With η-logic, the proof itself can be refined to mention that access is allowed at the time of the request. We make a formal connection between the two approaches in section 3.4.

In addition to its applications with PCA, we expect that η-logic can be used in specifying the behavior of systems with time-dependent authorization policies. In such cases, the logic can be used to formally establish correctness properties of the system, as in [18]. The first example in section 4 illustrates this approach. Linearity plays a crucial role in this setting, facilitating accurate models of mutable state.

In the spirit of Gentzen's pioneering work on proof theory [20], we abstain from model-theoretic

semantics, instead presenting η-logic as a sequent calculus. This brings the logic closer to realization in tools like theorem provers and proof verifiers, and facilitates a detailed study of its meta-theory, which is central to our work. We establish several properties, including consistency and cut elimination, which increase confidence in the logic's foundations. Cut elimination also implies that the meanings of connectives in the logic are independent of each other. This makes the logic open to extension with new connectives. Establishing these properties is non-trivial, involving a deep interplay between inference rules and constraints.

In summary, this work makes several contributions. First, it introduces explicit time into reasoning about authorization. In contrast to existing approaches, this makes it possible to express and reason robustly about time-dependent policies.

Second, it formalizes implementations of PCA that deal with time in an extra-logical manner and rely only on validity intervals of embedded digitally signed certificates. We further show that policy enforcement in PCA (at least for a fragment) is no more difficult than in the logics that have been proposed previously.

Third, our system integrates explicit time and linearity. This represents a non-trivial challenge, because both time intervals and single-use assumptions restrict availability of hypotheses during reasoning, but in entirely different ways. Our meta-theorems, specifically the cut elimination and identity properties, show that these concepts are indeed compatible, at least in a constructive setting. The key is a novel combination of ideas from hybrid logic with constraints. The examples demonstrate that the combination allows logical expression and enforcement of a wide range of practically occurring policies which were previously intractable.

Related Work. Our work draws upon ideas from several kinds of logics. Most closely related are works on constructive authorization logic [18, 19], from which we borrow linearity, affirmation, and our style of presentation. The "says" construct in our logic was first introduced by Abadi *et al.* [2, 25], and adopted by almost all subsequent proposals.

The formalization of time in our presentation combines ideas from both hybrid logics [11, 33] and constraint-based logics [23, 34]. Such a combination has been studied to a limited extent in Temporal Annotated Constraint Logic Programming (TACLP) [17]. This work, done in the context of logic programming without authorizations, allows interval annotations on atomic formulas, similar to our $A @ I$ construct. Besides TACLP, we are unaware of any work that uses hybrid logic for modeling time.

Linearity, which is important for modeling consumable resources, was introduced in a logic by Girard [21]. The judgmental form of linear logic was first studied by Chang *et al.* [13]. The use of linearity in conjunction with authorization was first proposed by two of the present authors and others [18]. Some enforcement mechanisms in the distributed setting have also been described [10].

More broadly, this work relates to languages and logics for expressing and enforcing access control policies [1, 2, 7–9, 12, 14, 18, 19, 24–29, 32]. With the exception of the policy language Sec-PAL [7], we are not aware of explicit use of time in any of these proposals. SecPAL's enforcement of time is external to the language, based on a constraint system that is not reasoned about within the formal semantics. In contrast, our logic permits direct reasoning with validity of formulas. It would be interesting to study the potential formal connections between the two approaches.

The principal target of our design is proof-carrying authorization (PCA) [3–5]. In existing work on PCA, validity of certificates plays an integral part, but it is not included in the logic. Rather, it is enforced by a separate check. In η-logic, this check becomes part of proof-verification.

An alternate approach to reasoning about time is based on temporal logics [15]. Here, one reasons about events *relative* in time to others. Although sometimes useful in reasoning about security protocols, the approach appears to be ineffective in the context of security policies and PCA, which rely heavily on absolute time.

Another related line of work is interval temporal logic [31], where one reasons about sequences of states in an evolving system. Like temporal logic, the method seems inadequate for reasoning about authorization policies.

Organization of the Report. In section 2, we introduce our logic and its proof system, and study its meta-theory. Section 3 describes η-logic's application to PCA. Section 4 illustrates the expressiveness of the logic by showing examples that contain complex time-dependent relationships. Section 5 concludes the report.

2 η-logic: Authorization with Explicit Time

At its core, η-logic is a first-order intuitionistic logic. It integrates several other constructs: affirmations, linearity, hybrid worlds representing time, and constraints. While these constructs have been studied separately in the past, their interaction with each other is deep and non-trivial. In particular, the hybrid nature of η-logic interacts with all the other components, making it impossible to construct η-logic as an extension of either linear logic or a logic of affirmation without changing the nature of the underlying judgments.

Following Per Martin-Löf [30], we use a judgmental approach in describing the logic. We separate formulas from judgments, making the latter the objects of reasoning. In the interest of readability, we describe the logic in several steps. We begin by briefly describing the structure of first-order terms and sorts. Next, we describe the judgments that capture time, linearity, and affirmation. We then discuss constraints, and finally present the logic's connectives and proof rules.

2.1 First-order Terms and Sorts

We assume that the quantifiable terms can be typed into different sorts (denoted by the meta-variable s). We stipulate at least two sorts: a sort of principals (principal) and a sort of intervals of time (interval). If t is a term and Σ assigns sorts to all constants occurring in t, we write $\Sigma \vdash t{:}s$ to mean that the term t has sort s. We write $[t/x]A$ to denote the formula obtained by substituting the term t for all free occurrences of x in A.

Principals, denoted by the letter K, represent machines, users, or programs that make access requests or issue policies. Concretely, they may be simple bit strings that represent names, identifiers, or keys.

Intervals, denoted by the letter I, represent *sets of time points* over which formulas are true. Borrowing terminology from hybrid logic, they are worlds which qualify formulas. We do not fix structures for either time points or their sets, but postulate necessary conditions that must hold on them. These are described in section 2.3. Intuitively, one may think of time points as points on the real line, and sets I as closed intervals on the real line. However, it should be noted that the term interval is really a misnomer here; we could work with other kinds of sets as well. In many natural scenarios, the sets are intervals, and we therefore continue to use this nomenclature.

2.2 Judgments

Ordinarily, logic is concerned with the truth of formulas without reference to time. However, in access control, the truth of formulas changes with time. For instance, if the formula may_enter(Alice, Bob) means that Alice is allowed to enter Bob's office, then this formula may be true during Bob's office hours and untrue at other times.

Hence, in order to reason accurately about time in access control, the logic should reason about truth of formulas at specific times. This leads us to the basic judgment of our logic: "formula A is true at all time points in the set I," written $A[\![I]\!]$.

Following prior work on security logics [18], we would like to go a step further by adding linearity to the logic for modeling state and single-use authorizations. Accordingly, we add a second judgment: "formula A is true exactly once in the set I," written $A[I]$. This does not mean that A holds at exactly one time point in I, but rather that the authorization implied by A must be used at one time point in the interval.

For example, may_enter(Alice, Bob)$[\![I]\!]$ means that Alice may enter Bob's office any number of times during interval I, while may_enter(Alice, Bob)$[I]$ means that Alice must enter Bob's office exactly once during interval I.

Next, in order to allow reasoning from assumptions, a feature central to all logics, we introduce a *hypothetical judgment (sequent)*. It takes the following form:

$$\Sigma, \Psi; \Gamma; \Delta \Longrightarrow A[I]$$

$\Sigma, \Psi, \Gamma, \Delta$ have the syntax listed below:

$$
\begin{array}{lll}
\Sigma & ::= & \cdot \mid \Sigma, x{:}s \\
\Psi & ::= & \cdot \mid \Psi, I \supseteq I' \\
\Gamma & ::= & \cdot \mid \Gamma, A[\![I]\!] \\
\Delta & ::= & \cdot \mid \Delta, A[I]
\end{array}
$$

Σ assigns sorts to all first-order parameters occurring in the remaining sequent. Ψ records superset constraints on intervals mentioned in the formulas in the sequent. Γ contains assumptions that are true on specific intervals, and Δ represents assumptions that are true exactly once on specific intervals. Γ and Δ are often called *unrestricted hypotheses* and *linear hypotheses*, respectively.

The meaning of the entire sequent is: "For each solution to the constraints Ψ in the variables Σ we can prove that A is true exactly once during interval I, using each hypothesis in Δ exactly once and each hypothesis in Γ zero or more times."

The judgment $A[I]$ on the right side of $\Longrightarrow$ is often called the *consequent* of the sequent. Sequents cannot have consequents of the form $A[\![I]\!]$. This restriction is inherited from linear logic [13], but does not limit the expressiveness of the deductive system.

Affirmations. In order to model security policies issued by distinct principals, we need to reason about statements made by principals. We call such statements affirmations. Due to the hybrid nature of the logic, we have to associate time with affirmations. Accordingly, we introduce a new judgment: "during interval I it is true that principal K affirms that formula A is true," written $(K \text{ affirms } A) \text{ at } I$.

There are two important points here. First, the phrase "K affirms that formula A is true" is broadly construed: K may not directly state that A is true; instead, A may follow from other

statements that K has made. Second, I is the interval over which the *affirmation itself* is true, not K's intention of the interval on which A is true. If required, the latter may be encoded within A using the @ connective. For example, suppose that Bob creates the policy "Alice may enter Bob's office between 9 AM and 5 PM" and that this policy is valid from 2007 to 2008. Then, this fact is represented by the judgment $(\text{Bob affirms } (\text{may\_enter}(\text{Alice}, \text{Bob}) @ [9AM, 5PM]))$ at $[2007, 2008]$. Observe that here the interval I is $[2007, 2008]$, whereas the intended validity of the policy that Bob makes is $[9AM, 5PM]$.

Next, we add a new form of sequent to reason hypothetically about affirmations.

$$\Sigma, \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I$$

The meaning of this sequent is: "For each solution to the constraints Ψ in the variables Σ we can prove that K affirms A exactly once during interval I, using each hypothesis in Δ exactly once and each hypothesis in Γ zero or more times."

2.3   Constraints

Superset constraints of the form $I \supseteq I'$ are an integral part of η-logic. Formally, they are incorporated in the proof system using the following judgment:

$$\Sigma; \Psi \models I \supseteq I'$$

This judgment means that the constraints in Ψ entail that I is a superset of I', parametrically in the constants mentioned in Σ. We do not fix the exact rules governing this judgment because we do not stipulate a concrete structure for intervals. We expect that, in practice, this judgment would be implemented using a constraint solving procedure. The details of such a procedure would, of course, depend on the representation chosen for intervals. However, to obtain meta-theoretic results about the logic (section 2.5), we require the following properties. Here, C denotes arbitrary superset constraints.

1. (Hypothesis) $\Sigma; \Psi, C \models C$.
2. (Weakening) If $\Sigma; \Psi \models C$, then $\Sigma, \Sigma'; \Psi, \Psi' \models C$.
3. (Cut) If $\Sigma; \Psi \models C$ and $\Sigma; \Psi, C \models C'$, then $\Sigma; \Psi \models C'$.
4. (Substitution) If $\Sigma \vdash t{:}s$ and $\Sigma, x{:}s; \Psi \models C$, then $\Sigma; [t/x]\Psi \models [t/x]C$.
5. (Reflexivity) $\Sigma; \Psi \models I \supseteq I$.
6. (Transitivity) If $\Sigma; \Psi \models I \supseteq I'$ and $\Sigma; \Psi \models I' \supseteq I''$, then $\Sigma; \Psi \models I \supseteq I''$.

In the case where intervals are represented by closed intervals on the real line, such a constraint solver can be constructed in a straightforward manner.

2.4   Formulas and Proof Rules

Having described the basic judgments and constraints in η-logic, we now turn to the connectives allowed in formulas and the proof rules for sequents. We allow all connectives of intuitionistic linear logic, although, for the sake of brevity, we limit our discussion here to only a subset. (Rules for the remaining connectives can be found in Appendix A.) In addition, we introduce a new connective $A @ I$ to internalize the judgment $A[I]$ as a formula, include the connective $\langle K \rangle A$ (read "K says A") [18, 19] to internalize the affirmation judgment, and add the connective $I \supseteq I'$ to represent

superset constraints as formulas. Although we use the same notation $I \supseteq I'$ for denoting both formulas and constraints, this should not cause confusion since the intended meaning should be clear from the context.

The syntax of formulas is shown below. P denotes atomic formulas.

$$A, B \quad ::= \quad P \mid A \otimes B \mid A \supset B \mid A \multimap B \mid \forall x{:}s.A \mid A @ I \mid \langle K \rangle A \mid I \supseteq I'$$

$A \otimes B$ means that A and B are true simultaneously. We have two forms of implication: unrestricted $(A \supset B)$ and linear $(A \multimap B)$. They differ in that the pre-condition A in $A \supset B$ can be satisfied only if A can be established without the use of linear hypotheses, while there is no such restriction on the pre-condition of $A \multimap B$. Conversely, to prove $A \supset B$ we may use A arbitrarily many times to prove B, while A must be used exactly once in a proof of B to establish $A \multimap B$.

The proof rules for the sequent calculus are summarized in Figure 1. γ denotes an arbitrary consequent, either $A[I]$ or $(K$ affirms $A)$ at I. The meanings of connectives in η-logic are described entirely by these proof rules, without any additional semantics. This ensures that the intended reading of formulas coincides with the available formal proofs, which is desirable for PCA.

The init and copy rules capture the nature of linear and unrestricted hypotheses. If we assume that formula A is true once during interval I, and if $I \supseteq I'$, then we should certainly be able to conclude that A may be true once during the interval I'. For atomic formulas, this is captured by the init rule; for others, we prove it as a theorem (Theorem 2). The init rule also highlights the interaction between linearity, time, and constraints: its premise contains a constraint, and the fact that no other linear hypothesis besides $P[I]$ is allowed to occur captures linearity. The copy rule permits copying of an unrestricted hypothesis $A[\![I]\!]$ into the set of linear hypotheses. This may be repeated, thus allowing the unrestricted hypothesis to be used multiple times.

The remaining rules (with the exception of affirms) are related to the logic's connectives. Each rule is classified as either right or left, depending on whether it acts on the right side or the left side of $\Longrightarrow$. We start with the new connective: $A @ I$.

$A @ I$ captures the essence of the judgment $A[I]$ as a formula. This permits us to associate time intervals with formulas nested inside other formulas. The right rule @R means that $A[I]$ entails $A @ I[I']$. The left rule @L states that the assumption $A @ I[I']$ is stronger than the assumption $A[I]$. Together they imply that, as judgments, $A[I]$ and $A @ I[I']$ entail each other. This is intuitive: if a formula A is true during interval I, then this fact is true over all intervals I'. Or equivalently, once the truth of a formula has been qualified by an interval, a subsequent qualification is meaningless.

By its nature, interval containment is independent of time. Thus, we should be allowed to establish the judgment $I \supseteq I'[I'']$ whenever the *constraint* $I \supseteq I'$ holds. This is captured by the right rule, $\supseteq R$. Dually, if we assume the judgment $I \supseteq I'[I'']$, then we should also be justified in assuming that the constraint $I \supseteq I'$ holds. This is captured by the left rule, $\supseteq L$.

Next, we examine affirmation. The affirms rule relates affirmation to truth. It states that if it is provable that formula A is true during interval I, then it is provable that every principal affirms its truth during interval I. This is based on the idea that a proof is irrefutable evidence; if A has a proof, then every principal must be willing to affirm A.

The connective $\langle K \rangle A$ (read "K says A") internalizes affirmation as a formula. Its right rule $\langle \rangle R$ means that the judgment $(\langle K \rangle A)[I]$ holds whenever $(K$ affirms $A)$ at I holds. The left rule $\langle \rangle L$ means that if we are trying to establish that K affirms B during I', and we know both K says A during I and $I \supseteq I'$, then we are justified in assuming that A is true during I. This rule captures the idea that principals are accountable for their statements; having stated A, K cannot refute it,

Basic Rules

$$\frac{\Sigma;\Psi \models I \supseteq I' \quad (P \text{ atomic})}{\Sigma;\Psi;\Gamma;P[I] \Longrightarrow P[I']} \text{ init} \qquad \frac{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta,A[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta \Longrightarrow \gamma} \text{ copy}$$

$A @ I$

$$\frac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A @ I[I']} @R \qquad \frac{\Sigma;\Psi;\Gamma;\Delta,A[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta,A @ I[I'] \Longrightarrow \gamma} @L$$

$I \supseteq I'$

$$\frac{\Sigma;\Psi \models I \supseteq I'}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow I \supseteq I'[I'']} \supseteq R \qquad \frac{\Sigma;\Psi,I \supseteq I';\Gamma;\Delta \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta,I \supseteq I'[I''] \Longrightarrow \gamma} \supseteq L$$

Affirmation and $\langle K \rangle A$

$$\frac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \text{ affirms} \qquad \frac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \langle K \rangle A[I]} \langle\rangle R$$

$$\frac{\Sigma;\Psi;\Gamma;\Delta,A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I' \quad \Sigma;\Psi \models I \supseteq I'}{\Sigma;\Psi;\Gamma;\Delta,\langle K \rangle A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'} \langle\rangle L$$

Other Connectives

$$\frac{\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A[I] \quad \Sigma;\Psi;\Gamma;\Delta_2 \Longrightarrow B[I]}{\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2 \Longrightarrow A \otimes B[I]} \otimes R \qquad \frac{\Sigma;\Psi;\Gamma;\Delta,A[I],B[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta,A \otimes B[I] \Longrightarrow \gamma} \otimes L$$

$$\frac{\Sigma,i:\text{interval};\Psi,I \supseteq i;\Gamma;\Delta,A[i] \Longrightarrow B[i]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \multimap B[I]} \multimap R$$

$$\frac{\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A[I'] \quad \Sigma;\Psi \models I \supseteq I' \quad \Sigma;\Psi;\Gamma;\Delta_2,B[I'] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2,A \multimap B[I] \Longrightarrow \gamma} \multimap L$$

$$\frac{\Sigma,i:\text{interval};\Psi,I \supseteq i;\Gamma,A[\![i]\!];\Delta \Longrightarrow B[i]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \supset B[I]} \supset R$$

$$\frac{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow A[I'] \quad \Sigma;\Psi \models I \supseteq I' \quad \Sigma;\Psi;\Gamma;\Delta,B[I'] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta,A \supset B[I] \Longrightarrow \gamma} \supset L$$

$$\frac{\Sigma,a:s;\Psi;\Gamma;\Delta \Longrightarrow [a/x]A[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \forall x{:}s.A[I]} \forall R \qquad \frac{\Sigma;\Psi;\Gamma;\Delta,[t/x]A[I] \Longrightarrow \gamma \quad \Sigma \vdash t{:}s}{\Sigma;\Psi;\Gamma;\Delta,\forall x{:}s.A[I] \Longrightarrow \gamma} \forall L$$

Figure 1: Sequent calculus for η-logic

and hence, while reasoning about another affirmation by K, we can assume A. It is instructive to observe the interaction between time and affirmation in this rule.

Finally, we describe the rules for connectives borrowed from linear logic: $\otimes$, $\multimap$, $\supset$, and $\forall$. Although these rules may appear similar to corresponding rules in linear logic (without time), the meanings of the connectives must be reinterpreted because truth is always qualified with time in η-logic. The presence of time opens the possibility of choosing from many different kinds of rules, with each choice resulting in a different interaction between the connectives and @. For instance, our rules imply that @ distributes over $\otimes$ — that $(A \otimes B) @ I$ is equivalent to $(A @ I) \otimes (B @ I)$. However, this choice is not forced, and one may conceive logics that do not validate this equivalence. The proof rules shown here describe what we believe to be an elegant, useful, and simple possibility.

The right rule $\otimes R$ states that in order to show that $A \otimes B$ is true on I, it suffices to partition the linear hypotheses disjointly into two parts, using one part to establish that A holds on I and the other to show that B holds on I. The left rule $\otimes L$ is dual, stating that the assumption $A \otimes B$ on interval I is stronger than the pair of assumptions A, B, both on the interval I. Together with the rules for @, these rules imply the equivalence mentioned earlier.

The right rule $\multimap R$ means that in order to establish that $A \multimap B$ holds on interval I, it suffices to show that for every interval i such that $I \supseteq i$, $B[i]$ follows from the linear hypothesis $A[i]$. The left rule $\multimap L$ is dual, stating that if $A \multimap B$ is assumed to hold on I and A holds on any smaller interval I', then B holds on I'. Together these mean that $(A \multimap B) @ I$ represents a method of obtaining B from A on any subset of I.

The rule $\supset R$ is similar to $\multimap R$, except that in this case A is assumed to be unrestricted. Correspondingly, in the left rule $\supset L$, one must establish A without any linear hypotheses.

We can establish the formula $\forall x{:}s.A$ if we can establish $[a/x]A$ for every fresh constant a of sort s. This is captured by the right rule $\forall R$. The left rule $\forall L$ states that if we assume $\forall x{:}s.A$, then we can also assume $[t/x]A$ for any term t of sort s.

This completes our presentation of the proof rules of the sequent calculus. We now turn to the meta-theory of η-logic.

2.5   Meta-theory

Meta-theoretic properties are important for a logic of authorization because they not only provide assurance of a strong foundation for the logic, but are also useful in analysis of policies. Cut elimination, for example, implies that all proofs can be normalized, i.e., reduced to a canonical form. This canonical form often provides far more insight into the reasons why access was granted as compared to the original proof.

In our logic, meta-theoretic properties are important from yet another perspective. Since connectives are described entirely by the rules of the sequent calculus, it is absolutely essential that the basic meaning of hypothetical judgments (sequents) be respected by the rules. Formally, this is expressed by two properties: admissibility of cut and identity. Admissibility of cut states that if a judgment such as $A[I]$ can be established, and assuming this judgment, we can establish a second judgment, then the second judgment can be established directly. Identity states that whenever we assume a judgment, we can conclude it. We prove both properties for our logic. To establish the admissibility of cut, it must be stated in a more general form.

Theorem 1 (Admissibility of Cut).
  1. If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]$ and $\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow \gamma$, then $\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \gamma$.

2. *If* $\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A[I]$ *and* $\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow \gamma$, *then* $\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow \gamma$.

3. *If* $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K$ affirms $A)$ at I *and* $\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow (K$ affirms $C)$ at I' *and* $\Sigma; \Psi \models I \supseteq I'$, *then* $\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow (K$ affirms $C)$ at I'.

Proof. See Appendix B.3. $\qquad\square$

Theorem 2 (Identity). *For any proposition A, $\Sigma; \Psi; \Gamma; A[I] \Longrightarrow A[I']$ if $\Sigma; \Psi \models I \supseteq I'$.*

Proof. See Appendix B.1. $\qquad\square$

Cut elimination usually refers to the explicit elimination of cut as a rule of inference from the sequent calculus. It follows by a simple structural induction from the admissibility of cut, and is therefore omitted here.

In a hybrid logic like η-logic, we expect another important property: if we can establish $A[I]$, then we should be able to establish $A[I']$ for every subset I' of I. This property, called subsumption, is formally captured by the following theorem.

Theorem 3 (Subsumption). *If $\Sigma; \Psi \models I \supseteq I'$, then the following hold:*

1. *If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]$, then $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I']$.*
2. *If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K$ affirms $A)$ at I, then $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K$ affirms $A)$ at I'.*

Proof. See Appendix B.2. $\qquad\square$

We now state some simple theorems that hold in the logic. Equally important are properties that cannot be established in their full generality. We write $\vdash A$ to mean $\cdot; \cdot; \cdot; \cdot \Longrightarrow A$ and $\nvdash A$ to mean that $\vdash A$ cannot be derived in full generality. Similarly $\models I \supseteq I'$ means that $\cdot; \cdot \models I \supseteq I'$.

1. $\nvdash ((A @ I) \multimap (A @ I'))[I'']$

2. $\vdash ((A @ I) \multimap (A @ I'))[I'']$ if $\models I \supseteq I'$

3. $\vdash ((A @ I @ I') \multimap (A @ I))[I'']$

4. $\vdash ((A @ I) \multimap (A @ I @ I'))[I'']$

5. $\vdash (((A \otimes B) @ I) \multimap ((A @ I) \otimes (B @ I)))[I']$

6. $\vdash (((A @ I) \otimes (B @ I)) \multimap ((A \otimes B) @ I))[I']$

7. $\nvdash A[I]$

The first property states that, in general, $A @ I$ does not imply $A @ I'$. In the special case where I is a superset of I', this is true (second property). The next two properties capture the nature of nested @ connectives: $A @ I @ I'$ and $A @ I$ are equivalent. Properties (5) and (6) imply that @ distributes over $\otimes$. The last property states consistency — not every formula is provable a priori in the logic.

The says connective $\langle K \rangle A$ is similar to a lax modality [16]. It satisfies the following theorems:

1. $\vdash (A \multimap \langle K \rangle A)[I]$

2. $\vdash ((\langle K \rangle \langle K \rangle A) \multimap \langle K \rangle A)[I]$

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3. $\vdash ((\langle K \rangle (A \multimap B)) \multimap ((\langle K \rangle A) \multimap (\langle K \rangle B)))[I]$

4. $\not\vdash ((\langle K \rangle A) \multimap A)[I]$

As a general design decision, we have kept the interaction between temporal constraints and logical reasoning as simple as possible. In particular, we do not permit splitting of intervals into sub-intervals during logical reasoning. For example, even if $I \cup I' = I''$ we can not prove in general that $A@I$ and $A@I'$ imply $A@I''$. For proof-carrying authorization (discussed in the next section), this means in order to demonstrate continuous right to access a resource over a given interval there must be a uniform proof over the whole interval, unless special policy axioms are introduced. The logic can easily be generalized to permit the splitting of intervals, but the theorem proving problem becomes significantly more difficult. Jia [23] provides an analysis of this trade-off in the setting of reasoning about imperative programs using a heap.

3 Proof-Carrying Authorization with η-logic

In this section, we describe applications of η-logic to PCA. The main merit of using η-logic for PCA is that the temporal validity of policies and credentials is reflected in the formulas of the logic, thus bringing the formalized policies closer to their intended meaning. We review the Grey system [5, 6] in section 3.1 and use it as an example to illustrate our PCA approach in section 3.2. In section 3.3, we comment on the feasibility of using η-logic in PCA. Finally, we formalize some of our claims about enforcement in section 3.4.

3.1 Review of the Grey System

The Grey system is an architecture for universal access control using proof-carrying authorization with smartphones. The Grey testbed is an implementation of keyless access control on office doors and computers, developed and currently deployed on one floor in the Collaborative Innovation Center at Carnegie Mellon University. Each office door is equipped with a processor that runs a proof-checking engine based on a logical framework. The processor controls an electronic relay which can unlock the door.

Enforcement of access control follows the standard PCA approach: a person desiring access to an office uses her cellphone to communicate with the office's door, sending it a proof that she is allowed access. This proof is checked by the proof-checking engine in the door, and if the proof is correct, the processor unlocks the door through the relay.

Two simple policies in Grey are the following:

1. A person may enter her own office.
2. A person may enter an office not belonging to her, if authorized to do so by the owner of the office.

In addition to policies, authorization in Grey relies on credentials issued by individual users that authorize other users to enter their offices. They are used in conjunction with the second policy. Physically, these credentials are digitally signed X.509 certificates. For pragmatic reasons, most of these credentials are time-bound: they are not valid forever because one usually does not want to allow another person to access her office indefinitely.

Each policy statement and each available credential is converted to a formula in Grey's logic. An individual wanting access must not only provide the door with a proof, but also any credentials used in the proof that the door may be unaware of. In addition to checking the proof, the door also checks the new credentials. If both checks succeed, the door opens. Otherwise, it does not.

Grey's current logic is oblivious to time. As a result, the validity bound on a credential is ignored when the credential is imported into the logic. For example, suppose Bob signs the credential "Allow Alice to enter my door (valid from 1/1/08 to 1/31/08)." If the predicate $\mathsf{may\_enter}(K_1, K_2)$ means that K_1 is allowed to enter K_2's office, then this credential may be imported into Grey's logic as the formula $\langle \mathrm{Bob} \rangle \mathsf{may\_enter}(\mathrm{Alice}, \mathrm{Bob})$. The validity bound of the credential is ignored in the logic. Policies are treated similarly — their validity, if any, is ignored. Consequently, proofs are ignorant of time, and it is possible to obtain a seemingly correct proof in the logic depending on formulas derived from expired credentials.

In order to rectify this problem and correctly enforce the time bounds in credentials, Grey uses an extra-logical mechanism. In addition to checking that a submitted proof and credentials are correct, a door also checks that all credentials used in the proof are valid *at the time of access*. Although secure and efficient in practice, this method divorces time from the logic, making reasoning in the logic inaccurate with respect to time. In particular, proof construction has to be augmented with a similar external time check. Otherwise, correct but expired proofs may be constructed. Furthermore, any meta-level analysis of the policies using the logic will be inaccurate with respect to time.

3.2 Grey in η-logic

In η-logic, we can model time-bounded credentials accurately. We illustrate this using policies from the Grey system. As before, let the predicate $\mathsf{may\_enter}(K_1, K_2)$ mean that K_1 is allowed to enter K_2's office. We assume the existence of an administrating principal, admin, who dictates all policies. For this example and all subsequent ones, we assume that time is represented by points on the real line, and intervals in the logic are intervals on the real line.

To open K_2's door at time t, K_1 must submit a proof showing that the following judgment is derivable from the available policies and credentials: $\langle \mathrm{admin} \rangle \mathsf{may\_enter}(K_1, K_2)[t, t]$. $[t, t]$ represents the closed point interval for time t. Observe that the judgment that must be established to gain access directly incorporates time. This is in sharp contrast to Grey's existing approach, where time is external to the logic.

Grey's policies described earlier can be imported as the following unrestricted hypotheses in η-logic.

1. $\langle \mathrm{admin} \rangle \forall K.\ \mathsf{may\_enter}(K, K) [\![(-\infty, \infty)]\!]$
2. $\langle \mathrm{admin} \rangle \forall K_1.\ \forall K_2.\ (\langle K_2 \rangle \mathsf{may\_enter}(K_1, K_2) \multimap \mathsf{may\_enter}(K_1, K_2)) [\![(-\infty, \infty)]\!]$

Here we have assumed that both policies are valid indefinitely, i.e., on the interval $(-\infty, \infty)$. If the policies are valid for only a finite duration of time, one may replace $(-\infty, \infty)$ with the appropriate interval.

A critical observation is that we assume that these formulas are unrestricted hypotheses because they may be used many times. This does not apply to credentials issued by individuals to allow others to enter their offices. For example, Bob may allow Alice to enter his office *once* between 1/1/08 and 1/31/08 by issuing a certificate that is imported as the linear hypothesis:

3. $\langle\text{Bob}\rangle\mathsf{may\_enter}(\text{Alice},\text{Bob})[1/1/08,1/31/08]$

It is instructive to check that, using the unrestricted hypotheses (1) and (2) and the linear hypothesis (3), it is possible to derive $\langle\text{admin}\rangle\mathsf{may\_enter}(\text{Alice},\text{Bob})[t,t]$ for any t in the time interval $[1/1/08,1/31/08]$. Also, it is impossible to derive the same judgment if t does not lie in this interval. Thus, qualifying formulas explicitly with intervals on which they are true makes proof construction in the logic accurate with respect to the time bounds on credentials.

3.3 Implementing PCA with η-logic

As described above, allowing explicit time in a logic bridges the gap between time-dependent credentials and their representation in the logic. The question then is whether this approach offers any advantages over traditional implementations of PCA.

The primary issue is efficiency. At first, one might think that adding time to the logic would slow proof-checking. While a comprehensive assessment of the efficiency of proof-checking can only be made with a real implementation, we show in section 3.4 that a reasonable fragment of the logic (namely, one in which there are no nested @ and $\supseteq$ connectives), can be implemented using the same method that Grey uses to enforce time-dependence of credentials: proof-checking and proof construction are done in oblivion to time, and validity of certificates at the time of access is ascertained separately. This fragment is large enough to express all policies of Grey, and other existing PCA based systems.

Thus, existing PCA systems can be implemented in η-logic without loss of efficiency. At the same time, there are several merits in making time explicit in the logic. First, policies and credentials are reflected more accurately in the logic. They therefore become amenable to more accurate meta-level policy analysis, such as an analysis for security loopholes. Second, leveraging the existing constraint-solving mechanism, one can model complex policies, policies that are intractable in previously proposed logics. Examples in section 4 include such policies. Third, with time-aware formulas, one cannot, even accidentally, construct a proof that is invalid due to a time-dependence. This reduces the risk of unanticipated access denials.

We anticipate new challenges if PCA is implemented using a fragment of η-logic larger than the one described above. An important issue that arises in proof search is certificate chain discovery: determining which credentials are relevant for a proof. In a time-aware logic, this problem is exacerbated, since this process has to incorporate temporal validity of certificates. However, there is a trade-off here: at the cost of more work, the final proof is guaranteed to be accurate. Alternatively, one may choose to ignore time during proof search. In that case, certificate chain discovery would revert to its usual complexity (and time-dependent inaccuracy).

An essential component that must be built into any realistic implementation of η-logic is a constraint solver. For simple constraints such as $I \supseteq I'$ that we have seen so far, this appears to be relatively straightforward. Furthermore, most policies arising in practice do not require parameters in constraints. This trivializes the constraint solving problem to checking containment over ground intervals. Even if one wished to be more ambitious by allowing other kinds of constraints for use in policies, previous work in constraint logic programming suggests that a large number of useful constraint domains are tractable in practice (see [22] for a survey).

An interesting, open problem in implementing PCA with η-logic is the treatment of linearity. Since linear hypotheses and the corresponding credentials must be consumed only once, a mechanism for tracking their use is required. If all linear credentials are maintained in a central database,

this is relatively straightforward. It is less clear, however, whether there is a uniform way of doing this in a completely distributed setting. Some initial ideas using contract signing protocols have been described earlier [10].

3.4 Enforcement for a Fragment of η-logic

The objective of this section is to show that Grey's method of checking credential validity at the time of request as a separate step after proof-checking can also be used for the fragment of η-logic without the connectives @ and $\supseteq$. This fragment does not preclude intervals in top-level judgments such as $A[I]$ and $A[\![I]\!]$. It covers all systems in which time is used only to bound the validity of credentials, but not inside the text of credentials, including all policies of the Grey system.

In order to formally describe our result we need a logic without time which is otherwise similar to η-logic. We choose the logic of [18], since our logic is derived from it. For the lack of a better name, we call this logic ζ-logic (ζ being the predecessor of η in the Greek alphabet). ζ-logic may be understood as the simplification of η-logic obtained by erasing intervals and constraints from formulas, judgments, sequents, and proof rules. The uninitiated reader may skip this section without affecting readability of the remaining report.

Let F denote formulas which do not contain the connectives @ and $\supseteq$. Such formulas are in the syntax of ζ-logic. Let Θ and Λ denote multisets of such formulas, representing unrestricted hypotheses and linear hypotheses in ζ-logic, respectively. Let $\vec{I}$ denote a list of ground intervals. Furthermore, if $\Theta = F_1, \ldots, F_n$, and $\vec{I} = I_1, \ldots, I_n$, let $\Theta[\![\vec{I}]\!]$ denote the set of unrestricted hypotheses $F_1[\![I_1]\!], \ldots, F_n[\![I_n]\!]$ in η-logic. Define $\Lambda[\vec{I'}]$ similarly. Also, let Ξ be the same as Σ except for the absence of interval parameters.

All sequents in this fragment of η-logic have one of the forms $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow F[I'']$ or $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } F) \text{ at } I''$.

Our idea for implementing PCA with this fragment of η-logic is the following. Whenever a principal needs to prove $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow F[I'']$, she instead proves that $\Sigma; \Theta; \Lambda \Longrightarrow F$ in ζ-logic. The proof checker verifies this proof in ζ-logic, and checks that each interval in $\vec{I}$ and $\vec{I'}$ is a superset of I''. As the following theorem shows, the success of these two checks implies that the original sequent is provable in η-logic.

(A priori, this result was not obvious to us because intervals mentioned in the last sequent of a proof interact with subformulas in other sequents of the proof. It seemed entirely possible that some subtle consequence of these interactions would not be captured by simply checking that each interval in $\vec{I}$ and $\vec{I'}$ is a superset of I''.)

Theorem 4. *Suppose* $\Sigma; \Psi \models I''' \supseteq I''$ *for each* $I''' \in \vec{I}$ *and for each* $I''' \in \vec{I'}$. *Then,*

1. *If* $\Xi; \Theta; \Lambda \Longrightarrow F$ *in* ζ-*logic, then* $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow F[I'']$ *in* η-*logic.*
2. *If* $\Xi; \Theta; \Lambda \Longrightarrow K$ affirms F *in* ζ-*logic, then* $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K$ affirms $F) $ at I'' *in* η-*logic.*

Proof. See Appendix C.1. □

Thus, on the fragment without @ and $\supseteq$, proof-checking in a logic without time, together with simple containment checking for intervals *soundly* approximates proof-checking in η-logic. One might also expect the converse to hold, namely that whenever $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow F[I'']$ holds in η-logic, $\Xi; \Theta; \Lambda \Longrightarrow F$ holds in ζ-logic and for each interval $I''' \in \vec{I}$ and each $I''' \in \vec{I'}$, $\Sigma; \Psi \models I''' \supseteq I''$. This is partially correct: given that the η-logic sequent is provable, the former holds as the following

theorem shows, but the latter may not. The reason is quite straightforward: the consequent of the sequent may not depend on some assumptions in Θ, and the intervals associated with such assumptions may have no relation to I''.

Theorem 5.

1. *If* $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow F[I'']$, *then* $\Xi; \Theta; \Lambda \Longrightarrow F$ *in* ζ-*logic.*
2. *If* $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } F)$ at I'', *then* $\Xi; \Theta; \Lambda \Longrightarrow K$ affirms F *in* ζ-*logic.*

Proof. See Appendix C.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Expressiveness of η-logic: More Examples

Besides modeling time-bounded credentials, η-logic, through its combination of explicit time and constraints, can also be used to express very complicated policies. We illustrate this expressiveness through two hypothetical examples. The first example describes the policies of a homework assignment administration system at a university. In addition to time, this example uses linearity to model changes of state. The second example describes the policies of a peer review publication process.

A Homework Assignment Administration System. We consider the policy of a hypothetical homework administration system in a university. Time is used to explicitly encode the release and due dates of each assignment. The policies allow professors to create assignments for the courses they teach and adjust their release and due dates. Students can view an assignment *after* the release date and submit it *before* the due date. Modeling this policy creates complex interactions between time and authorization that cannot be captured without either a connective like @ or constraints.

We use the meta-variable A to denote assignments, C for courses, P for professors, and S for students. The predicates (with their intuitive meanings) and policies used in this example are summarized in Figure 2. As a syntactic convention, we assume that $\otimes$, $\multimap$, and $\supset$ are right associative and that the binding precedences are, in decreasing order: $\langle\rangle$; @; $\otimes$; $\multimap$ and $\supset$; $\forall$. We write $t \in I$ as an abbreviation for $I \supseteq [t, t]$, and $t \geq t'$ as an abbreviation for $t \in [t', \infty)$.

As may be expected, all policy rules are unrestricted hypotheses that are valid forever. This is indicated by the annotation $[\![(-\infty, \infty)]\!]$ on each policy rule.

We assume an administrating principal, admin. At the beginning of each semester, this principal issues credentials to students registered for courses and professors teaching courses. These must be presented later (perhaps many times) to view, submit, and change assignments. As a result, they are unrestricted hypotheses. They have the logical forms $\langle\text{admin}\rangle\text{is\_student}(S, C)[\![Sem]\!]$ and $\langle\text{admin}\rangle\text{is\_professor}(P, C)[\![Sem]\!]$ respectively, where Sem denotes the semester under consideration.

A professor P can create an assignment A in a course C by issuing a credential stating $\langle P\rangle\text{is\_assignment}(A, C)[t_r, t_d]$. The time points t_r and t_d stand for the release and due dates of the assignment, respectively. $[t_r, t_d]$ denotes the closed interval between these time points. We require that such credentials be linear hypotheses. If instead they were unrestricted, then there would be no logical mechanism to change the release and due dates after creating an assignment.

To view an assignment A in course C at time t, a student S must be able to prove the judgment $\langle\text{admin}\rangle\text{may\_view}(S, A, C)[t, t]$. The policy rule named view allows students to do this. We assume

$\mathsf{request\_view}(A, C)$ A request to view assignment A of course C.

$\mathsf{request\_submit}(A, C)$ A request to submit answers for assignment A of course C.

$\mathsf{is\_professor}(P, C)$ P is a professor for course C.

$\mathsf{is\_student}(S, C)$ S is a student enrolled in course C.

$\mathsf{is\_assignment}(A, C)$ A is an assignment for the students in course C.

$\mathsf{may\_view}(S, A, C)$ S may view assignment A of course C.

$\mathsf{may\_submit}(S, A, C)$ S may submit answers for assignment A of course C.

$\mathsf{change\_date}(A, C, t'_r, t'_d)$ A request to change the release and due dates for assignment A of course C to t'_r and t'_d, respectively.

Policies

$$\mathsf{view} : (\langle S \rangle \mathsf{request\_view}(A, C) @ [t, t] \multimap$$
$$\langle \mathrm{admin} \rangle \mathsf{is\_student}(S, C) @ [t, t] \supset$$
$$\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t_r, t_d] \multimap$$
$$\langle \mathrm{admin} \rangle \mathsf{is\_professor}(P, C) @ [t_r, t_d] \supset$$
$$(t \geq t_r) \supset$$
$$\langle \mathrm{admin} \rangle \mathsf{may\_view}(S, A, C) @ [t, t] \otimes$$
$$\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t_r, t_d]) [\![(-\infty, \infty)]\!]$$

$$\mathsf{submit} : (\langle S \rangle \mathsf{request\_submit}(A, C) @ [t, t] \multimap$$
$$\langle \mathrm{admin} \rangle \mathsf{is\_student}(S, C) @ [t, t] \supset$$
$$\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t_r, t_d] \multimap$$
$$\langle \mathrm{admin} \rangle \mathsf{is\_professor}(P, C) @ [t_r, t_d] \supset$$
$$(t \in [t_r, t_d]) \supset$$
$$\langle \mathrm{admin} \rangle \mathsf{may\_submit}(S, A, C) @ [t, t] \otimes$$
$$\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t_r, t_d]) [\![(-\infty, \infty)]\!]$$

$$\mathsf{change} : (\langle P \rangle \mathsf{change\_date}(A, C, t'_r, t'_d) \multimap$$
$$\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t_r, t_d] \multimap$$
$$\langle \mathrm{admin} \rangle \mathsf{is\_professor}(P, C) \supset$$
$$\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t'_r, t'_d]) [\![(-\infty, \infty)]\!]$$

Figure 2: Predicates and policies for a homework assignment administration system

an implicit universal quantification over the variables S, A, C, t, P, t_r, and t_d. Intuitively, this rule states that a student S may view an assignment A in course C at time t by issuing a credential $\langle S \rangle \mathsf{request\_view}(A, C)$ valid at the time of request, $[t, t]$, if the following can be established:

1. $\langle \mathrm{admin} \rangle \mathsf{is\_student}(S, C) @ [t, t]$, i.e., the student is registered for the course at the time of request. To establish this, the student must use the credential she received from admin at the beginning of the semester.

2. $\langle P \rangle \mathsf{is\_assignment}(A, C) @ [t_r, t_d]$, i.e., a professor P states that A is an assignment of course C with release date t_r and due date t_d.

3. $\langle\text{admin}\rangle$is_professor$(P, C)$ @ $[t_r, t_d]$, i.e., P is a professor teaching the course C for the entire duration of the assignment. This can be established using the credential issued by admin to the professor.

4. $t \geq t_r$, i.e., the time of request is after the release of the assignment. This preempts attempts to read the assignment before it is officially released.

If each of these four conditions are satisfied, then the student may view the assignment. There are two important observations to be made here. First, the linear hypothesis $\langle P\rangle$is_assignment(A, C) @ $[t_r, t_d]$ consumed in condition 2 is regenerated at the end. Second, explicit time is crucial for modeling the constraint $t \geq t_r$. Such a policy rule cannot be modeled using only time bounds on credentials.

Similarly, the submit policy rule allows a student S to submit an assignment between its release and due dates by issuing a credential of the form $\langle S\rangle$request_submit$(A, C)[t, t]$. In this case the objective is to establish that $\langle\text{admin}\rangle$may_submit$(S, A, C)$ @ $[t, t]$, where t is the time at which the submission is made.

Our final policy rule, change, illustrates the use of linearity in modeling change of state. It allows a professor P to change the release and due dates of an assignment A in a course C he is teaching by issuing the credential $\langle P\rangle$change_date(A, C, t'_r, t'_d), where t'_r and t'_d are the new release and due dates of the assignment. The policy *consumes* the earlier hypothesis defining the release and due dates of the assignment and replaces it with a new one. For this to work properly, it is essential that such hypotheses be linear, not unrestricted. Failure to ensure this would result in two hypotheses defining the release and due dates of the same assignment after application of the rule.

A Peer Review Publication Process. We further illustrate the expressiveness of our logic by describing the policies of a hypothetical peer review and publication process of an academic journal. This example differs slightly from the previous example in that the policies are not fixed. Instead, they are created by principals using templates.

We use the meta-variable A to range over articles considered for publication, R and K for reviewers, J for journals, and E for editors. The predicates and policies used in this example are summarized in Figure 3. An important point to observe is that the policies are not issued by fixed principals; instead, each editor and each journal issues credentials containing the policies.

We stipulate that each journal J appoint an editor E during time period I by issuing the credential $\langle J\rangle$is_editor$(E, J)[\![I]\!]$. The editor E can then declare R a reviewer for article A from time t onward by issuing the credential $\langle E\rangle$is_reviewer$(R, A, J)[\![[t, \infty)]\!]$.

In addition, E can start accepting reviews by issuing a credential that establishes the accept policy. While issuing the credential, the editor should instantiate I_E to the interval over which reviews may be accepted. All variables other than E and I_E are assumed to be universally quantified. Once established, the policy allows an appointed reviewer R to submit a review on article A at time t_a by signing the credential $\langle R\rangle$is_approved$(A, R, J)[t_a, t_a]$. If $t_a \in I_E$ the policy can be used to conclude that the editor considers the article approved.

In an analogous manner, each journal J can establish a publishing policy by issuing a credential following the form of publish. In issuing this credential, I_J should be instantiated to the interval during which articles are accepted for publication. All variables other than J and I_J are assumed to be universally quantified. Once established, the policy states that if an editor E says at time t_a that an article A has been approved, and t_a is in I_J, then the article is considered published from

$\boxed{\text{Predicates}}$

is_approved(A, K, J) Article A is approved by principal K for publication in journal J.
is_reviewer(R, A, J) R is the reviewer for article A submitted to journal J.
is_editor(E, J) E is an editor for journal J.
is_published(A, J) Article A is published in journal J.

$\boxed{\text{Policies}}$

$$\text{approve} : \langle E \rangle (\langle R \rangle \text{is\_approved}(A, R, J) @ [t_a, t_a] \multimap$$
$$\text{is\_reviewer}(R, A, J) @ [t_a, t_a] \supset$$
$$(t_a \in I_E) \supset$$
$$\text{is\_approved}(A, E, J) @ [t_a, \infty)) [\![(-\infty, \infty)]\!]$$

$$\text{publish} : \langle J \rangle (\langle E \rangle \text{is\_approved}(A, E, J) @ [t_a, t_a'] \multimap$$
$$\text{is\_editor}(E, J) @ [t_a, t_a] \supset$$
$$(t_a \in I_J) \supset$$
$$\text{is\_published}(A, J) @ [t_a, \infty)) [\![(-\infty, \infty)]\!]$$

Figure 3: Predicates and policies for a peer review publication process

time t_a onward.

5   Conclusion

This report has presented a logic that combines time, linearity, hybrid worlds, and authorization in a novel way. Our proof-theoretic approach resulted in a clean meta-theory. Among other properties, we established cut elimination. We also showed that a reasonably expressive fragment of our logic can be enforced in a PCA architecture in a straightforward manner. Through examples, we illustrated the expressiveness of the logic and demonstrated scenarios which cannot be modeled in earlier proposals.

An important topic that remains open is the analysis of policies written in the logic. We expect that work from prior logics, particularly non-interference theorems [19], will carry over to η-logic. It will be interesting to study how these theorems interact with time.

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A Complete η-logic Sequent Calculus

The complete set of inference rules of η-logic are summarized below. We extend the formulas of the logic to include all of the standard formulas of intuitionistic linear logic, with the exception of additive falsehood, $\mathbf{0}$. $\mathbf{0}$ is not included in the logic because we believe it to be a security risk: from the assumption $\mathbf{0} @ I'$, one can conclude any fact. The syntax of formulas is:

$$A, B \quad ::= \quad P \mid A \otimes B \mid \mathbf{1} \mid A \,\&\, B \mid \top \mid A \oplus B \mid A \multimap B \mid A \supset B \mid \forall x{:}s.A \mid A @ I \mid \langle K \rangle A \mid I \sqsupseteq I'$$

The new formulas do not interact with time in interesting ways: they follow the pattern by which the $\otimes$ connective interacts with time. The proofs given in subsequent appendices include these formulas.

$\boxed{\text{Basic Rules}}$

$$\frac{\Sigma; \Psi \models I \sqsupseteq I' \quad (P \text{ atomic})}{\Sigma; \Psi; \Gamma; P[I] \Longrightarrow P[I']} \text{ init} \qquad \frac{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta, A[I] \Longrightarrow \gamma}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta \Longrightarrow \gamma} \text{ copy}$$

$\boxed{A @ I}$

$$\frac{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A @ I[I']} @R \qquad \frac{\Sigma; \Psi; \Gamma; \Delta, A[I] \Longrightarrow \gamma}{\Sigma; \Psi; \Gamma; \Delta, A @ I[I'] \Longrightarrow \gamma} @L$$

$\boxed{I \sqsupseteq I'}$

$$\frac{\Sigma; \Psi \models I \sqsupseteq I'}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow I \sqsupseteq I'[I'']} \sqsupseteq R \qquad \frac{\Sigma; \Psi, I \sqsupseteq I'; \Gamma; \Delta \Longrightarrow \gamma}{\Sigma; \Psi; \Gamma; \Delta, I \sqsupseteq I'[I''] \Longrightarrow \gamma} \sqsupseteq L$$

$\boxed{\text{Affirmation and } \langle K \rangle A}$

$$\frac{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \text{ affirms} \qquad \frac{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \langle K \rangle A[I]} \langle \rangle R$$

$$\frac{\Sigma; \Psi; \Gamma; \Delta, A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I' \quad \Sigma; \Psi \models I \sqsupseteq I'}{\Sigma; \Psi; \Gamma; \Delta, \langle K \rangle A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'} \langle \rangle L$$

$\boxed{\text{Other Connectives}}$

$$\dfrac{\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A[I] \quad \Sigma;\Psi;\Gamma;\Delta_2 \Longrightarrow B[I]}{\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2 \Longrightarrow A \otimes B[I]} \ \otimes R \qquad\qquad \dfrac{\Sigma;\Psi;\Gamma;\Delta, A[I], B[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, A \otimes B[I] \Longrightarrow \gamma} \ \otimes L$$

$$\dfrac{}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow \mathbf{1}[I]} \ \mathbf{1}R \qquad\qquad \dfrac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, \mathbf{1}[I] \Longrightarrow \gamma} \ \mathbf{1}L$$

$$\dfrac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A[I] \quad \Sigma;\Psi;\Gamma;\Delta \Longrightarrow B[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \,\&\, B[I]} \ \&R$$

$$\dfrac{\Sigma;\Psi;\Gamma;\Delta, A[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, A \,\&\, B[I] \Longrightarrow \gamma} \ \&L_1 \qquad\qquad \dfrac{\Sigma;\Psi;\Gamma;\Delta, B[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, A \,\&\, B[I] \Longrightarrow \gamma} \ \&L_2$$

$$\dfrac{}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \top[I]} \ \top R$$

$$\dfrac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \oplus B[I]} \ \oplus R_1 \qquad\qquad \dfrac{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow B[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \oplus B[I]} \ \oplus R_2$$

$$\dfrac{\Sigma;\Psi;\Gamma;\Delta, A[I] \Longrightarrow \gamma \quad \Sigma;\Psi;\Gamma;\Delta, B[I] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, A \oplus B[I] \Longrightarrow \gamma} \ \oplus L$$

$$\dfrac{\Sigma, i\text{:interval};\Psi, I \supseteq i;\Gamma;\Delta, A[i] \Longrightarrow B[i]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \multimap B[I]} \ \multimap R$$

$$\dfrac{\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A[I'] \quad \Sigma;\Psi \models I \supseteq I' \quad \Sigma;\Psi;\Gamma;\Delta_2, B[I'] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2, A \multimap B[I] \Longrightarrow \gamma} \ \multimap L$$

$$\dfrac{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow A[I]}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow \,!A[I]} \ !R \qquad\qquad \dfrac{\Sigma;\Psi;\Gamma, A[\![I]\!];\Delta \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, !A[I] \Longrightarrow \gamma} \ !L$$

$$\dfrac{\Sigma, i\text{:interval};\Psi, I \supseteq i;\Gamma, A[\![i]\!];\Delta \Longrightarrow B[i]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A \supset B[I]} \ \supset R$$

$$\dfrac{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow A[I'] \quad \Sigma;\Psi \models I \supseteq I' \quad \Sigma;\Psi;\Gamma;\Delta, B[I'] \Longrightarrow \gamma}{\Sigma;\Psi;\Gamma;\Delta, A \supset B[I] \Longrightarrow \gamma} \ \supset L$$

$$\dfrac{\Sigma, a{:}s;\Psi;\Gamma;\Delta \Longrightarrow [a/x]A[I]}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \forall x{:}s.A[I]} \ \forall R \qquad\qquad \dfrac{\Sigma;\Psi;\Gamma;\Delta, [t/x]A[I] \Longrightarrow \gamma \quad \Sigma \vdash t{:}s}{\Sigma;\Psi;\Gamma;\Delta, \forall x{:}s.A[I] \Longrightarrow \gamma} \ \forall L$$

B   Meta-theoretic Proofs

B.1   Identity Principle

Theorem 2. If $\Sigma;\Psi \models I \supseteq I'$, then $\Sigma;\Psi;\Gamma;A[I] \Longrightarrow A[I']$.

Proof. By structural induction on the proposition A. We name the given derivation $\mathcal{D}$.

Case: $A = P$

$$\Sigma; \Psi; \Gamma; P[I] \Longrightarrow P[I'] \hspace{8em} \text{init Rule on } \mathcal{D}$$

Case: $A = A_1 \otimes A_2$

$\Sigma; \Psi; \Gamma; A_1[I] \Longrightarrow A_1[I']$	I.H. on A_1 and $\mathcal{D}$
$\Sigma; \Psi; \Gamma; A_2[I] \Longrightarrow A_2[I']$	I.H. on A_2 and $\mathcal{D}$
$\Sigma; \Psi; \Gamma; A_1[I], A_2[I] \Longrightarrow A_1 \otimes A_2[I']$	$\otimes R$ Rule on previous lines
$\Sigma; \Psi; \Gamma; A_1 \otimes A_2[I] \Longrightarrow A_1 \otimes A_2[I']$	$\otimes L$ Rule on previous line

Case: $A = \mathbf{1}$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow \mathbf{1}[I']$	$\mathbf{1}R$ Rule
$\Sigma; \Psi; \Gamma; \mathbf{1}[I] \Longrightarrow \mathbf{1}[I']$	$\mathbf{1}L$ Rule on previous line

Case: $A = A_1 \,\&\, A_2$

$\Sigma; \Psi; \Gamma; A_1[I] \Longrightarrow A_1[I']$	I.H. on A_1 and $\mathcal{D}$
$\Sigma; \Psi; \Gamma; A_1 \,\&\, A_2[I] \Longrightarrow A_1[I']$	$\&L_1$ Rule on previous line
$\Sigma; \Psi; \Gamma; A_2[I] \Longrightarrow A_2[I']$	I.H. on A_2 and $\mathcal{D}$
$\Sigma; \Psi; \Gamma; A_1 \,\&\, A_2[I] \Longrightarrow A_2[I']$	$\&L_2$ Rule on previous line
$\Sigma; \Psi; \Gamma; A_1 \,\&\, A_2[I] \Longrightarrow A_1 \,\&\, A_2[I']$	$\&R$ Rule on second and fourth lines

Case: $A = \top$

$$\Sigma; \Psi; \Gamma; \top[I] \Longrightarrow \top[I'] \hspace{8em} \top R \text{ Rule}$$

Case: $A = A_1 \oplus A_2$

$\Sigma; \Psi; \Gamma; A_1[I] \Longrightarrow A_1[I']$	I.H. on A_1 and $\mathcal{D}$
$\Sigma; \Psi; \Gamma; A_1[I] \Longrightarrow A_1 \oplus A_2[I']$	$\oplus R_1$ Rule on previous line
$\Sigma; \Psi; \Gamma; A_2[I] \Longrightarrow A_2[I']$	I.H. on A_2 and $\mathcal{D}$
$\Sigma; \Psi; \Gamma; A_2[I] \Longrightarrow A_1 \oplus A_2[I']$	$\oplus R_2$ Rule on previous line
$\Sigma; \Psi; \Gamma; A_1 \oplus A_2[I] \Longrightarrow A_1 \oplus A_2[I']$	$\oplus L$ Rule on second and fourth lines

Case: $A = A_1 \multimap A_2$

$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i' \models i' \supseteq i'$	Reflexivity Property of $\models$
$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; A_1[i'] \Longrightarrow A_1[i']$	I.H. on A_1 and previous line
$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; A_2[i'] \Longrightarrow A_2[i']$	I.H. on A_2 and first line
$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i' \models I \supseteq I'$	Weakening Property of $\models$ on $\mathcal{D}$
$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i' \models I' \supseteq i'$	Hypothesis Property of $\models$
$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i' \models I \supseteq i'$	Transitivity Property of $\models$ on fourth and fifth lines
$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; A_1 \multimap A_2[I], A_1[i'] \Longrightarrow A_2[i']$	$\multimap L$ Rule on second, sixth, and third lines
$\Sigma; \Psi; \Gamma; A_1 \multimap A_2[I] \Longrightarrow A_1 \multimap A_2[I']$	$\multimap R$ Rule on previous line

Case: $A = !A_1$

$$\Sigma; \Psi; \Gamma, A_1\llbracket I \rrbracket; A_1[I] \Longrightarrow A_1[I'] \qquad\qquad \text{I.H. on } A_1 \text{ and } \mathcal{D}$$
$$\Sigma; \Psi; \Gamma, A_1\llbracket I \rrbracket; \cdot \Longrightarrow A_1[I'] \qquad\qquad \text{copy Rule on previous line}$$
$$\Sigma; \Psi; \Gamma, A_1\llbracket I \rrbracket; \cdot \Longrightarrow !A_1[I'] \qquad\qquad !R \text{ Rule on previous line}$$
$$\Sigma; \Psi; \Gamma; !A_1[I] \Longrightarrow !A_1[I'] \qquad\qquad !L \text{ Rule on previous line}$$

Case: $A = A_1 \supset A_2$

$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i' \models i' \supseteq i' \qquad\qquad \text{Reflexivity Property of } \models$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i'; \Gamma, A_1\llbracket i' \rrbracket; A_1[i'] \Longrightarrow A_1[i'] \qquad\qquad \text{I.H. on } A_1 \text{ and previous line}$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i'; \Gamma, A_1\llbracket i' \rrbracket; \cdot \Longrightarrow A_1[i'] \qquad\qquad \text{copy Rule on previous line}$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i'; \Gamma, A_1\llbracket i' \rrbracket; A_2[i'] \Longrightarrow A_2[i'] \qquad\qquad \text{I.H. on } A_2 \text{ and first line}$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i' \models I \supseteq I' \qquad\qquad \text{Weakening Property of } \models \text{ on } \mathcal{D}$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i' \models I' \supseteq i' \qquad\qquad \text{Hypothesis Property of } \models$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i' \models I \supseteq i' \qquad\qquad \text{Transitivity Property of } \models \text{ on fifth and sixth lines}$$
$$\Sigma, i'{:}\text{interval}; \Psi, I' \supseteq i'; \Gamma, A_1\llbracket i' \rrbracket; A_1 \supset A_2[I] \Longrightarrow A_2[i'] \qquad\qquad \supset L \text{ Rule on third, seventh,}$$
$$\text{and fourth lines}$$
$$\Sigma; \Psi; \Gamma; A_1 \supset A_2[I] \Longrightarrow A_1 \supset A_2[I'] \qquad\qquad \supset R \text{ Rule on previous line}$$

Case: $A = \forall x{:}s.A_1$

$$\Sigma, x{:}s \vdash x{:}s$$
$$\Sigma, x{:}s; \Psi; \Gamma; A_1[I] \Longrightarrow A_1[I'] \qquad\qquad \text{I.H. on } A_1 \text{ and } \mathcal{D}$$
$$\Sigma, x{:}s; \Psi; \Gamma; \forall x{:}s.A_1[I] \Longrightarrow A_1[I'] \qquad\qquad \forall L \text{ Rule on previous lines}$$
$$\Sigma; \Psi; \Gamma; \forall x{:}s.A_1[I] \Longrightarrow \forall x{:}s.A_1[I'] \qquad\qquad \forall R \text{ Rule on previous line}$$

Case: $A = A_1 @ I''$

$$\Sigma; \Psi \Longrightarrow I'' \supseteq I'' \qquad\qquad \text{Reflexivity Property of } \models$$
$$\Sigma; \Psi; \Gamma; A_1[I''] \Longrightarrow A_1[I''] \qquad\qquad \text{I.H. on } A_1 \text{ and previous line}$$
$$\Sigma; \Psi; \Gamma; A_1[I''] \Longrightarrow A_1 @ I''[I'] \qquad\qquad @R \text{ Rule on previous line}$$
$$\Sigma; \Psi; \Gamma; A_1 @ I''[I] \Longrightarrow A_1 @ I''[I'] \qquad\qquad @L \text{ Rule on previous line}$$

Case: $A = \langle K \rangle A_1$

$$\Sigma; \Psi; \Gamma; A_1[I] \Longrightarrow A_1[I'] \qquad\qquad \text{I.H. on } A_1 \text{ and } \mathcal{D}$$
$$\Sigma; \Psi; \Gamma; A_1[I] \Longrightarrow (K \text{ affirms } A_1) \text{ at } I' \qquad\qquad \text{affirms Rule on previous line}$$
$$\Sigma; \Psi; \Gamma; \langle K \rangle A_1[I] \Longrightarrow (K \text{ affirms } A_1) \text{ at } I' \qquad\qquad \langle \rangle L \text{ Rule on previous line and } \mathcal{D}$$
$$\Sigma; \Psi; \Gamma; \langle K \rangle A_1[I] \Longrightarrow \langle K \rangle A_1[I'] \qquad\qquad \langle \rangle R \text{ Rule on previous line}$$

Case: $A = I_2 \supseteq I_3$

$$\Sigma; \Psi, I_2 \supseteq I_3 \models I_2 \supseteq I_3 \qquad\qquad \text{Hypothesis Property of } \models$$
$$\Sigma; \Psi, I_2 \supseteq I_3; \Gamma; \cdot \Longrightarrow I_2 \supseteq I_3[I'] \qquad\qquad \supseteq R \text{ Rule on previous line}$$
$$\Sigma; \Psi; \Gamma; I_2 \supseteq I_3[I] \Longrightarrow I_2 \supseteq I_3[I'] \qquad\qquad \supseteq L \text{ Rule on previous line}$$

$\square$

B.2 Subsumption

Before we can prove the subsumption theorem, we must prove two lemmata.

B.2.1 Transitivity for Constraint Hypotheses in the Constraint Domain

Lemma 1. If $\Sigma; \Psi, I \supseteq I'' \models C$ and $\Sigma; \Psi \models I \supseteq I'$, then $\Sigma; \Psi, I' \supseteq I'' \models C$.

Proof. Let $\mathcal{D} = \Sigma; \Psi, I \supseteq I'' \models C$ and $\mathcal{E} = \Sigma; \Psi \models I \supseteq I'$.

$\Sigma; \Psi, I' \supseteq I'' \models I \supseteq I'$	Weakening Property of $\models$ on $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I'' \models I' \supseteq I''$	Hypothesis Property of $\models$
$\Sigma; \Psi, I' \supseteq I'' \models I \supseteq I''$	Transitivity Property of $\models$ on previous lines
$\Sigma; \Psi, I \supseteq I'', I' \supseteq I'' \models C$	Weakening Property of $\models$ on $\mathcal{D}$
$\Sigma; \Psi, I' \supseteq I'' \models C$	Cut Property of $\models$ on third and fourth lines

$\square$

B.2.2 Transitivity for Constraint Hypotheses

Lemma 2. If $\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow \gamma$ and $\Sigma; \Psi \models I \supseteq I'$, then $\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow \gamma$.

Proof. By structural induction on the first given derivation.

Case:

$$\mathcal{D} = \cfrac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi, I \supseteq I'' \models I_3 \supseteq I_4}}{\Sigma; \Psi, I \supseteq I''; \Gamma; P[I_3] \Longrightarrow P[I_4]} \; \text{init}$$

$\Sigma; \Psi, I' \supseteq I'' \models I_3 \supseteq I_4$	Lemma 1 on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; P[I_3] \Longrightarrow P[I_4]$	init Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma', A[\![I_3]\!]; \Delta, A[I_3] \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma', A[\![I_3]\!]; \Delta \Longrightarrow \gamma} \; \text{copy}$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma', A[\![I_3]\!]; \Delta, A[I_3] \Longrightarrow \gamma$	I.H. on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma', A[\![I_3]\!]; \Delta \Longrightarrow \gamma$	copy Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1 \Longrightarrow A_1[I_3]} \quad \overset{\displaystyle \mathcal{D}_2}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_2 \Longrightarrow A_2[I_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2[I_3]} \ \otimes R$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1 \Longrightarrow A_1[I_3]$	I.H. on $\mathcal{D}_1$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_2 \Longrightarrow A_2[I_3]$	I.H. on $\mathcal{D}_2$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2[I_3]$	$\otimes R$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1[I_3], A_2[I_3] \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1 \otimes A_2[I_3] \Longrightarrow \gamma} \ \otimes L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1[I_3], A_2[I_3] \Longrightarrow \gamma$	I.H. on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1 \otimes A_2[I_3] \Longrightarrow \gamma$	$\otimes L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{}{\Sigma; \Psi, I \supseteq I''; \Gamma; \cdot \Longrightarrow \mathbf{1}[I_3]} \ \mathbf{1}R$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \cdot \Longrightarrow \mathbf{1}[I_3]$	$\mathbf{1}R$ Rule

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1 \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, \mathbf{1}[I_3] \Longrightarrow \gamma} \ \mathbf{1}L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1 \Longrightarrow \gamma$	I.H. on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, \mathbf{1}[I_3] \Longrightarrow \gamma$	$\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1[I_3]} \quad \overset{\displaystyle \mathcal{D}_2}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_2[I_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \mathbin{\&} A_2[I_3]} \ \mathbin{\&}R$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1[I_3]$	I.H. on $\mathcal{D}_1$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_2[I_3]$	I.H. on $\mathcal{D}_2$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \mathbin{\&} A_2[I_3]$	$\mathbin{\&}R$ Rule on previous lines

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1[I_3] \Longrightarrow \gamma\end{array}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1 \,\&\, A_2[I_3] \Longrightarrow \gamma} \ \&L_1$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1[I_3] \Longrightarrow \gamma$	I.H. on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1 \,\&\, A_2[I_3] \Longrightarrow \gamma$	$\&L_1$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_2[I_3] \Longrightarrow \gamma\end{array}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1 \,\&\, A_2[I_3] \Longrightarrow \gamma} \ \&L_2$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_2[I_3] \Longrightarrow \gamma$	I.H. on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1 \,\&\, A_2[I_3] \Longrightarrow \gamma$	$\&L_2$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow \top[I_3]} \ \top R$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow \top[I_3]$	$\top R$ Rule

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1[I_3]\end{array}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I_3]} \ \oplus R_1$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1[I_3]$	I.H. on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I_3]$	$\oplus R_1$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_2[I_3]\end{array}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I_3]} \ \oplus R_2$$

$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_2[I_3] \qquad\qquad \text{I.H. on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I_3] \qquad\qquad \oplus R_2 \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1[I_3] \Longrightarrow \gamma} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_2[I_3] \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1 \oplus A_2[I_3] \Longrightarrow \gamma} \ \oplus L$$

$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1[I_3] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D}_1 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_2[I_3] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D}_2 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1 \oplus A_2[I_3] \Longrightarrow \gamma \qquad\qquad \oplus L \text{ Rule on previous lines}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma, i_3\text{:interval}; \Psi, I \supseteq I'', I_3 \supseteq i_3; \Gamma; \Delta, A_1[i_3] \Longrightarrow A_2[i_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I_3]} \ \multimap R$$

$$\Sigma, i_3\text{:interval}; \Psi, I_3 \supseteq i_3 \models I \supseteq I' \qquad\qquad \text{Weakening Property of } \models \text{ on } \mathcal{E}$$
$$\Sigma, i_3\text{:interval}; \Psi, I' \supseteq I'', I_3 \supseteq i_3; \Gamma; \Delta, A_1[i_3] \Longrightarrow A_2[i_3] \qquad\qquad \text{I.H. on } \mathcal{D}' \text{ and previous line}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I_3] \qquad\qquad \multimap R \text{ Rule on previous line}$$

Case: The last rule in $\mathcal{D}$ is $\multimap L$, and $\mathcal{D}$ has the form:

$$\cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1 \Longrightarrow A_1[I_4]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi, I \supseteq I'' \models I_3 \supseteq I_4} \quad \overset{\mathcal{D}_3}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_2, A_2[I_4] \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2[I_3] \Longrightarrow \gamma} \ \multimap L$$

$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1 \Longrightarrow A_1[I_4] \qquad\qquad \text{I.H. on } \mathcal{D}_1 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I'' \models I_3 \supseteq I_4 \qquad\qquad \text{Lemma 1 on } \mathcal{D}_2 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_2, A_2[I_4] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D}_3 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2[I_3] \Longrightarrow \gamma \qquad\qquad \multimap L \text{ Rule on previous lines}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \cdot \Longrightarrow A[I_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \cdot \Longrightarrow \,!A[I_3]} \ !R$$

$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \cdot \Longrightarrow A[I_3] \qquad\qquad \text{I.H. on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \cdot \Longrightarrow !A[I_3] \qquad\qquad !R \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma, A[\![I_3]\!]; \Delta_1 \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, !A[I_3] \Longrightarrow \gamma} \; !L$$

$$\Sigma; \Psi, I' \supseteq I''; \Gamma, A[\![I_3]\!]; \Delta_1 \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, !A[I_3] \Longrightarrow \gamma \qquad\qquad !L \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}'}{\Sigma, i_3\text{:interval}; \Psi, I \supseteq I'', I_3 \supseteq i_3; \Gamma, A_1[\![i_3]\!]; \Delta \Longrightarrow A_2[i_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I_3]} \; \supset R$$

$$\Sigma, i_3\text{:interval}; \Psi, I_3 \supseteq i_3 \models I \supseteq I' \qquad\qquad \text{Weakening Property of } \models \text{ on } \mathcal{E}$$
$$\Sigma, i_3\text{:interval}; \Psi, I' \supseteq I'', I_3 \supseteq i_3; \Gamma, A_1[\![i_3]\!]; \Delta \Longrightarrow A_2[i_3] \qquad\qquad \text{I.H. on } \mathcal{D}' \text{ and previous line}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I_3] \qquad\qquad \supset R \text{ Rule on previous line}$$

Case: The last rule in $\mathcal{D}$ is $\supset L$, and $\mathcal{D}$ has the form:

$$\frac{\overset{\textstyle \mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \cdot \Longrightarrow A_1[I_4]} \quad \overset{\textstyle \mathcal{D}_2}{\Sigma; \Psi, I \supseteq I'' \models I_3 \supseteq I_4} \quad \overset{\textstyle \mathcal{D}_3}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_2[I_4] \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A_1 \supset A_2[I_3] \Longrightarrow \gamma} \; \supset L$$

$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \cdot \Longrightarrow A_1[I_4] \qquad\qquad \text{I.H. on } \mathcal{D}_1 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I'' \models I_3 \supseteq I_4 \qquad\qquad \text{Lemma 1 on } \mathcal{D}_2 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_2[I_4] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D}_3 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A_1 \supset A_2[I_3] \Longrightarrow \gamma \qquad\qquad \supset L \text{ Rule on previous lines}$$

Case:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A[I_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow \forall x{:}s.A[I_3]} \; \forall R$$

$$\Sigma, x{:}s; \Psi \models I \supseteq I' \qquad\qquad \text{Weakening Property of } \models \text{ on } \mathcal{E}$$
$$\Sigma, x{:}s; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A[I_3] \qquad\qquad \text{I.H. on } \mathcal{D}' \text{ and previous line}$$
$$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow \forall x{:}s.A[I_3] \qquad\qquad \forall R \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, [t/x]A[I_3] \Longrightarrow \gamma} \quad \overset{\mathcal{D}_2}{\Sigma \vdash t{:}s}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, \forall x{:}s.A[I_3] \Longrightarrow \gamma} \; \forall L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, [t/x]A[I_3] \Longrightarrow \gamma$ I.H. on $\mathcal{D}_1$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, \forall x{:}s.A[I_3] \Longrightarrow \gamma$ $\forall L$ Rule on previous line and $\mathcal{D}_2$

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A[I_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A \,@\, I_3[I_4]} \; @R$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A[I_3]$ I.H. on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A \,@\, I_3[I_4]$ $@R$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A[I_3] \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A \,@\, I_3[I_4] \Longrightarrow \gamma} \; @L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A[I_3] \Longrightarrow \gamma$ I.H. on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A \,@\, I_3[I_4] \Longrightarrow \gamma$ $@L$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow A[I_3]}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I_3} \; \text{affirms}$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A[I_3]$ I.H. on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I_3$ affirms Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I_3}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta \Longrightarrow \langle K \rangle A[I_3]} \; \langle\rangle R$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I_3$ I.H. on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow \langle K \rangle A[I_3]$ $\langle \rangle R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, A[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I_4} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi, I \supseteq I'' \models I_3 \supseteq I_4}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, \langle K \rangle A[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I_4} \; \langle \rangle L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, A[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I_4$ I.H. on $\mathcal{D}_1$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I'' \models I_3 \supseteq I_4$ Lemma 1 on $\mathcal{D}_2$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, \langle K \rangle A[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I_4$ $\langle \rangle L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I'' \models I_3 \supseteq I_4}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \cdot \Longrightarrow I_3 \supseteq I_4[I_5]} \; \supseteq R$$

$\Sigma; \Psi, I' \supseteq I'' \models I_3 \supseteq I_4$ Lemma 1 on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \cdot \Longrightarrow I_3 \supseteq I_4[I_5]$ $\supseteq R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi, I \supseteq I'', I_3 \supseteq I_4; \Gamma; \Delta_1 \Longrightarrow \gamma}}{\Sigma; \Psi, I \supseteq I''; \Gamma; \Delta_1, I_3 \supseteq I_4[I_5] \Longrightarrow \gamma} \; \supseteq L$$

$\Sigma; \Psi, I_3 \supseteq I_4 \models I \supseteq I'$ Weakening Property of $\models$ on $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I'', I_3 \supseteq I_4; \Gamma; \Delta_1 \Longrightarrow \gamma$ I.H. on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1, I_3 \supseteq I_4[I_5] \Longrightarrow \gamma$ $\supseteq L$ Rule on previous line

$\square$

B.2.3 Subsumption Proof

Theorem 3.
1. If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]$ and $\Sigma; \Psi \models I \supseteq I'$, then $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I']$.
2. If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I$ and $\Sigma; \Psi \models I \supseteq I'$, then $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I'$.

Proof. By simultaneous structural induction on the first given derivation, $\mathcal{D}$.
     Part 1:

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi \models I'' \supseteq I\end{array}}{\Sigma;\Psi;\Gamma;P[I''] \Longrightarrow P[I]}\ \text{init}$$

$\Sigma;\Psi \Longrightarrow I'' \supseteq I'$ Transitivity Property of $\models$ on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;P[I''] \Longrightarrow P[I']$ init Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta,B[I''] \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta \Longrightarrow A[I]}\ \text{copy}$$

$\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta,B[I''] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta \Longrightarrow A[I']$ copy Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A_1[I] & \Sigma;\Psi;\Gamma;\Delta_2 \Longrightarrow A_2[I]\end{array}}{\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2 \Longrightarrow A_1 \otimes A_2[I]}\ \otimes R$$

$\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A_1[I']$ I.H.(1) on $\mathcal{D}_1$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta_2 \Longrightarrow A_2[I']$ I.H.(1) on $\mathcal{D}_2$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2 \Longrightarrow A_1 \otimes A_2[I']$ $\otimes R$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma;\Delta_1,B_1[I''],B_2[I''] \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma;\Delta_1,B_1 \otimes B_2[I''] \Longrightarrow A[I]}\ \otimes L$$

$\Sigma;\Psi;\Gamma;\Delta_1,B_1[I''],B_2[I''] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta_1,B_1 \otimes B_2[I''] \Longrightarrow A[I']$ $\otimes L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow \mathbf{1}[I]}\ \mathbf{1}R$$

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$$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow \mathbf{1}[I'] \qquad\qquad \mathbf{1}R \text{ Rule}$$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow A[I]\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, \mathbf{1}[I''] \Longrightarrow A[I]} \ \mathbf{1}L$$

$\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, \mathbf{1}[I''] \Longrightarrow A[I']$ $\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I] & \Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I]\end{array}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \ \& \ A_2[I]} \ \&R$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I']$ I.H.(1) on $\mathcal{D}_1$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I']$ I.H.(1) on $\mathcal{D}_2$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \ \& \ A_2[I']$ $\&R$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow A[I]\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \ \& \ B_2[I''] \Longrightarrow A[I]} \ \&L_1$$

$\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \ \& \ B_2[I''] \Longrightarrow A[I']$ $\&L_1$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow A[I]\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \ \& \ B_2[I''] \Longrightarrow A[I]} \ \&L_2$$

$\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \ \& \ B_2[I''] \Longrightarrow A[I']$ $\&L_2$ Rule on previous line

Case:

$$\mathcal{D} = \frac{}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \top[I]} \ \top R$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \top[I']$ $\top R$ Rule

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_1$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I']$ $\oplus R_1$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_2$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I']$ $\oplus R_2$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow A[I]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \oplus B_2[I''] \Longrightarrow A[I]} \ \oplus L$$

$\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}_1$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}_2$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \oplus B_2[I''] \Longrightarrow A[I']$ $\oplus L$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma, i{:}\mathsf{interval}; \Psi, I \supseteq i; \Gamma; \Delta, A_1[i] \Longrightarrow A_2[i]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I]} \ \multimap R$$

$\Sigma, i{:}\mathsf{interval}; \Psi, I' \supseteq i; \Gamma; \Delta, A_1[i] \Longrightarrow A_2[i]$ Lemma 2 on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I']$ $\multimap R$ Rule on previous line

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Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow B_1[I_3]} \quad \overset{\displaystyle \mathcal{D}_2}{\Sigma; \Psi \models I_2 \supseteq I_3} \quad \overset{\displaystyle \mathcal{D}_3}{\Sigma; \Psi; \Gamma; \Delta_2, B_2[I_3] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2[I_2] \Longrightarrow A[I]} \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta_2, B_2[I_3] \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}_3$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2[I_2] \Longrightarrow A[I']$

         $\multimap L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A_1[I]}}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow !A_1[I]} \; !R$$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A_1[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow !A_1[I']$ $!R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi; \Gamma, B[\![I'']\!]; \Delta_1 \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, !B[I''] \Longrightarrow A[I]} \; !L$$

$\Sigma; \Psi; \Gamma, B[\![I'']\!]; \Delta_1 \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta_1, !B[I''] \Longrightarrow A[I']$ $!L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma, i{:}\mathsf{interval}; \Psi, I \supseteq i; \Gamma, A_1[\![i]\!]; \Delta \Longrightarrow A_2[i]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I]} \; \supset R$$

$\Sigma, i{:}\mathsf{interval}; \Psi, I' \supseteq i; \Gamma, A_1[\![i]\!]; \Delta \Longrightarrow A_2[i]$ Lemma 2 on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I']$ $\supset R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}_1}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow B_1[I_3]} \quad \overset{\displaystyle \mathcal{D}_2}{\Sigma; \Psi \models I_2 \supseteq I_3} \quad \overset{\displaystyle \mathcal{D}_3}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I_3] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \supset B_2[I_2] \Longrightarrow A[I]} \supset L$$

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$$\Sigma; \Psi; \Gamma; \Delta_1, B_2[I_3] \Longrightarrow A[I'] \qquad\qquad \text{I.H.(1) on } \mathcal{D}_3 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \supset B_2[I_2] \Longrightarrow A[I'] \qquad\qquad \supset L \text{ Rule on } \mathcal{D}_1, \mathcal{D}_2, \text{ and previous line}$$

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \forall x{:}s.A[I]} \ \forall R$$

$$\Sigma, x{:}s; \Psi \models I \supseteq I' \qquad\qquad \text{Weakening Property of } \models \text{ on } \mathcal{E}$$
$$\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A[I'] \qquad\qquad \text{I.H.(1) on } \mathcal{D}' \text{ and previous line}$$
$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \forall x{:}s.A[I'] \qquad\qquad \forall R \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta, [t/x]B[I''] \Longrightarrow A[I]} \quad \overset{\displaystyle \mathcal{D}_2}{\Sigma \vdash t{:}s}}{\Sigma; \Psi; \Gamma; \Delta, \forall x{:}s.B[I''] \Longrightarrow A[I]} \ \forall L$$

$$\Sigma; \Psi; \Gamma; \Delta, [t/x]B[I''] \Longrightarrow A[I'] \qquad\qquad \text{I.H.(1) on } \mathcal{D}_1 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta, \forall x{:}s.B[I''] \Longrightarrow A[I'] \qquad\qquad \forall L \text{ Rule on previous line and } \mathcal{D}_2$$

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I'']}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 @ I''[I]} \ @R$$

$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 @ I''[I'] \qquad\qquad @R \text{ Rule on } \mathcal{D}'$$

Case:

$$\mathcal{D} = \frac{\overset{\displaystyle \mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta_1, B[I_3] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, B @ I_3[I_2] \Longrightarrow A[I]} \ @L$$

$$\Sigma; \Psi; \Gamma; \Delta_1, B[I_3] \Longrightarrow A[I'] \qquad\qquad \text{I.H.(1) on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B @ I_3[I_2] \Longrightarrow A[I'] \qquad\qquad @L \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow (K \text{ affirms } A_1) \text{ at } I}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \langle K \rangle A_1[I]} \ \langle\rangle R$$

$\Sigma;\Psi;\Gamma;\Delta \Longrightarrow (K \text{ affirms } A_1) \text{ at } I'$ I.H.(2) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \langle K \rangle A_1[I']$ $\langle\rangle R$ Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma;\Psi \models I'' \supseteq I'''}}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow I'' \supseteq I'''[I]} \ \supseteq R$$

$\Sigma;\Psi;\Gamma;\cdot \Longrightarrow I'' \supseteq I'''[I']$ $\supseteq R$ Rule on $\mathcal{D}'$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma;\Psi,I_2 \supseteq I_3;\Gamma;\Delta_1 \Longrightarrow A[I]}}{\Sigma;\Psi;\Gamma;\Delta_1,I_2 \supseteq I_3[I_4] \Longrightarrow A[I]} \ \supseteq L$$

$\Sigma;\Psi,I_2 \supseteq I_3 \models I \supseteq I'$ Weakening Property of $\models$ on $\mathcal{E}$
$\Sigma;\Psi,I_2 \supseteq I_3;\Gamma;\Delta_1 \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and previous line
$\Sigma;\Psi;\Gamma;\Delta_1,I_2 \supseteq I_3[I_4] \Longrightarrow A[I']$ $\supseteq L$ Rule on previous line

Part 2:

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta,B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \text{copy}$$

$\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta,B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ I.H.(2) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma;\Psi;\Gamma',B[\![I'']\!];\Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ copy Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma;\Psi;\Gamma;\Delta_1,B_1[I''],B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma;\Psi;\Gamma;\Delta_1,B_1 \otimes B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \otimes L$$

$$\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''], B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \text{I.H.(2) on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \otimes B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \otimes L \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, \mathbf{1}[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \mathbf{1}L$$

$$\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \text{I.H.(2) on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, \mathbf{1}[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \mathbf{1}L \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \,\&\, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \& L_1$$

$$\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \text{I.H.(2) on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \,\&\, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \& L_1 \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \,\&\, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \& L_2$$

$$\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \text{I.H.(2) on } \mathcal{D}' \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \,\&\, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \& L_2 \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \quad \cfrac{\mathcal{D}_2}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \oplus B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \oplus L$$

$$\Sigma; \Psi; \Gamma; \Delta_1, B_1[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \text{I.H.(2) on } \mathcal{D}_1 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \text{I.H.(2) on } \mathcal{D}_2 \text{ and } \mathcal{E}$$
$$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \oplus B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I' \qquad \oplus L \text{ Rule on previous lines}$$

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow B_1[I_3]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I_2 \supseteq I_3} \quad \overset{\mathcal{D}_3}{\Sigma; \Psi; \Gamma; \Delta_2, B_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta_2, B_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{2cm}$ I.H.(2) on $\mathcal{D}_3$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{0.5cm}$ $\multimap L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma, B[\![I'']\!]; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, !B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ !L$$

$\Sigma; \Psi; \Gamma, B[\![I'']\!]; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{2cm}$ I.H.(2) on $\mathcal{D}'$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, !B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{2cm}$ $!L$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow B_1[I_3]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I_2 \supseteq I_3} \quad \overset{\mathcal{D}_3}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \supset B_2[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \supset L$$

$\Sigma; \Psi; \Gamma; \Delta_1, B_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{2cm}$ I.H.(2) on $\mathcal{D}_3$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, B_1 \supset B_2[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{0.5cm}$ $\supset L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta, [t/x]B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \quad \overset{\mathcal{D}_2}{\Sigma \vdash t{:}s}}{\Sigma; \Psi; \Gamma; \Delta, \forall x{:}s.B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \forall L$$

$\Sigma; \Psi; \Gamma; \Delta, [t/x]B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{2cm}$ I.H.(2) on $\mathcal{D}_1$ and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta, \forall x{:}s.B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\hspace{1cm}$ $\forall L$ Rule on previous line and $\mathcal{D}_2$

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta_1, B[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, B @ I_3[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ @L$$

$\Sigma; \Psi; \Gamma; \Delta_1, B[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ I.H.(2) on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta_1, B @ I_3[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ @L Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1, B[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I_2 \supseteq I}}{\Sigma; \Psi; \Gamma; \Delta_1, \langle K \rangle B[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \langle\rangle L$$

$\Sigma; \Psi; \Gamma; \Delta_1, B[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ I.H.(2) on $\mathcal{D}_1$ and $\mathcal{E}$

$\Sigma; \Psi \models I_2 \supseteq I'$ Transitivity Property of $\models$ on $\mathcal{D}_2$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta_1, \langle K \rangle B[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\langle\rangle L$ Rule on previous lines

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \text{affirms}$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I']$ I.H.(1) on $\mathcal{D}'$ and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ affirms Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi, I_2 \supseteq I_3; \Gamma; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, I_2 \supseteq I_3[I_4] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \supseteq L$$

$\Sigma; \Psi, I_2 \supseteq I_3 \models I \supseteq I'$ Weakening Property of $\models$ on $\mathcal{E}$

$\Sigma; \Psi, I_2 \supseteq I_3; \Gamma; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ I.H.(2) on $\mathcal{D}'$ and previous line

$\Sigma; \Psi; \Gamma; \Delta_1, I_2 \supseteq I_3[I_4] \Longrightarrow (K \text{ affirms } A) \text{ at } I'$ $\supseteq L$ Rule on previous line

$\square$

B.3   Admissibility of Cut

Before we can prove the admissibility of cut, we must prove a few lemmas.

B.3.1 Constraint Cut Lemma

Lemma 3. If $\Sigma; \Psi \models C$ and $\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow \gamma$, then $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \gamma$.

Proof. By structural induction on the second given derivation, $\mathcal{E}$.

Case:

$$\mathcal{E} = \frac{\begin{array}{c} \mathcal{E}' \\ \Sigma; \Psi, C \models I \supseteq I' \end{array}}{\Sigma; \Psi, C; \Gamma; P[I] \Longrightarrow P[I']} \ \text{init}$$

$\Sigma; \Psi \Longrightarrow I \supseteq I'$ Cut Property of $\models$ on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; P[I] \Longrightarrow P[I']$ init Rule on previous line

Case:

$$\mathcal{E} = \frac{\begin{array}{c} \mathcal{E}' \\ \Sigma; \Psi, C; \Gamma', A[\![I]\!]; \Delta, A[I] \Longrightarrow \gamma \end{array}}{\Sigma; \Psi, C; \Gamma', A[\![I]\!]; \Delta \Longrightarrow \gamma} \ \text{copy}$$

$\Sigma; \Psi; \Gamma', A[\![I]\!]; \Delta, A[I] \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma', A[\![I]\!]; \Delta \Longrightarrow \gamma$ copy Rule on previous line

Case:

$$\mathcal{E} = \frac{\begin{array}{cc} \mathcal{E}_1 & \mathcal{E}_2 \\ \Sigma; \Psi, C; \Gamma; \Delta_1 \Longrightarrow A_1[I] & \Sigma; \Psi, C; \Gamma; \Delta_2 \Longrightarrow A_2[I] \end{array}}{\Sigma; \Psi, C; \Gamma; \Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2[I]} \ \otimes R$$

$\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow A_1[I]$ I.H. on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma; \Psi; \Gamma; \Delta_2 \Longrightarrow A_2[I]$ I.H. on $\mathcal{D}$ and $\mathcal{E}_2$
$\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2[I]$ $\otimes R$ Rule on previous lines

Case:

$$\mathcal{E} = \frac{\begin{array}{c} \mathcal{E}' \\ \Sigma; \Psi, C; \Gamma; \Delta_1, A_1[I], A_2[I] \Longrightarrow \gamma \end{array}}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1 \otimes A_2[I] \Longrightarrow \gamma} \ \otimes L$$

$\Sigma; \Psi; \Gamma; \Delta_1, A_1[I], A_2[I] \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1, A_1 \otimes A_2[I] \Longrightarrow \gamma$ $\otimes L$ Rule on previous line

Case:

$$\mathcal{E} = \dfrac{}{\Sigma; \Psi, C; \Gamma; \cdot \Longrightarrow \mathbf{1}[I]} \; \mathbf{1}R$$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow \mathbf{1}[I]$ $\mathbf{1}R$ Rule

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi, C; \Gamma; \Delta_1 \Longrightarrow \gamma}}{\Sigma; \Psi, C; \Gamma; \Delta_1, \mathbf{1}[I] \Longrightarrow \gamma} \; \mathbf{1}L$$

$\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1, \mathbf{1}[I] \Longrightarrow \gamma$ $\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1[I]} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_2[I]}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1 \,\&\, A_2[I]} \; \&R$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I]$ I.H. on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I]$ I.H. on $\mathcal{D}$ and $\mathcal{E}_2$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \,\&\, A_2[I]$ $\&R$ Rule on previous lines

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1[I] \Longrightarrow \gamma}}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1 \,\&\, A_2[I] \Longrightarrow \gamma} \; \&L_1$$

$\Sigma; \Psi; \Gamma; \Delta_1, A_1[I] \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1, A_1 \,\&\, A_2[I] \Longrightarrow \gamma$ $\&L_1$ Rule on previous line

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_2[I] \Longrightarrow \gamma}}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1 \,\&\, A_2[I] \Longrightarrow \gamma} \; \&L_2$$

$$\Sigma; \Psi; \Gamma; \Delta_1, A_2[I] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D} \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta_1, A_1 \mathbin{\&} A_2[I] \Longrightarrow \gamma \qquad\qquad \&L_2 \text{ Rule on previous line}$$

Case:

$$\mathcal{E} = \dfrac{}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow \top[I]} \ \top R$$

$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \top[I] \qquad\qquad \top R \text{ Rule}$$

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1[I]}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_1$$

$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I] \qquad\qquad \text{I.H. on } \mathcal{D} \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I] \qquad\qquad \oplus R_1 \text{ Rule on previous line}$$

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_2[I]}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_2$$

$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I] \qquad\qquad \text{I.H. on } \mathcal{D} \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I] \qquad\qquad \oplus R_2 \text{ Rule on previous line}$$

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1[I] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_2[I] \Longrightarrow \gamma}}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1 \oplus A_2[I] \Longrightarrow \gamma} \ \oplus L$$

$$\Sigma; \Psi; \Gamma; \Delta_1, A_1[I] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D} \text{ and } \mathcal{E}_1$$
$$\Sigma; \Psi; \Gamma; \Delta_1, A_2[I] \Longrightarrow \gamma \qquad\qquad \text{I.H. on } \mathcal{D} \text{ and } \mathcal{E}_2$$
$$\Sigma; \Psi; \Gamma; \Delta_1, A_1 \oplus A_2[I] \Longrightarrow \gamma \qquad\qquad \oplus L \text{ Rule on previous lines}$$

Case:

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}'\\ \Sigma, i\text{:interval}; \Psi, C, I \supseteq i; \Gamma; \Delta, A_1[i] \Longrightarrow A_2[i]\end{array}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I]} \ \multimap R$$

$\Sigma, i\text{:interval}; \Psi, I \supseteq i \models C$	Weakening Property of $\models$ on $\mathcal{D}$
$\Sigma, i\text{:interval}; \Psi, I \supseteq i; \Gamma; \Delta, A_1[i] \Longrightarrow A_2[i]$	I.H. on previous line and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I]$	$\multimap R$ Rule on previous line

Case:

$$\mathcal{E} = \cfrac{\begin{array}{ccc}\mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3\\ \Sigma; \Psi, C; \Gamma; \Delta_1 \Longrightarrow A_1[I'] & \Sigma; \Psi, C \models I \supseteq I' & \Sigma; \Psi, C; \Gamma; \Delta_2, A_2[I'] \Longrightarrow \gamma\end{array}}{\Sigma; \Psi, C; \Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2[I] \Longrightarrow \gamma} \ \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow A_1[I']$	I.H. on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma; \Psi \models I \supseteq I'$	Cut Property of $\models$ on $\mathcal{D}$ and $\mathcal{E}_2$
$\Sigma; \Psi; \Gamma; \Delta_2, A_2[I'] \Longrightarrow \gamma$	I.H. on $\mathcal{D}$ and $\mathcal{E}_3$
$\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2[I] \Longrightarrow \gamma$	$\multimap L$ Rule on previous lines

Case:

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}'\\ \Sigma; \Psi, C; \Gamma; \cdot \Longrightarrow A[I]\end{array}}{\Sigma; \Psi, C; \Gamma; \cdot \Longrightarrow !A[I]} \ !R$$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A[I]$	I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow !A[I]$	$!R$ Rule on previous line

Case:

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}'\\ \Sigma; \Psi, C; \Gamma, A[\![I]\!]; \Delta_1 \Longrightarrow \gamma\end{array}}{\Sigma; \Psi, C; \Gamma; \Delta_1, !A[I] \Longrightarrow \gamma} \ !L$$

$\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1 \Longrightarrow \gamma$	I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1, !A[I] \Longrightarrow \gamma$	$!L$ Rule on previous line

Case:

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}'\\ \Sigma, i\text{:interval}; \Psi, C, I \supseteq i; \Gamma, A_1[\![i]\!]; \Delta \Longrightarrow A_2[i]\end{array}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I]} \ \supset R$$

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$\Sigma, i{:}\mathsf{interval}; \Psi, I \supseteq i \models C$ Weakening Property of $\models$ on $\mathcal{D}$

$\Sigma, i{:}\mathsf{interval}; \Psi, I \supseteq i; \Gamma, A_1[\![i]\!]; \Delta \Longrightarrow A_2[i]$ I.H. on previous line and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I]$ $\supset R$ Rule on previous line

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi, C; \Gamma; \cdot \Longrightarrow A_1[I']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi, C \models I \supseteq I'} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_2[I'] \Longrightarrow \gamma}}{\Sigma; \Psi, C; \Gamma; \Delta_1, A_1 \supset A_2[I] \Longrightarrow \gamma} \supset L$$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A_1[I']$ I.H. on $\mathcal{D}$ and $\mathcal{E}_1$

$\Sigma; \Psi \models I \supseteq I'$ Cut Property of $\models$ on $\mathcal{D}$ and $\mathcal{E}_2$

$\Sigma; \Psi; \Gamma; \Delta_1, A_2[I'] \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}_3$

$\Sigma; \Psi; \Gamma; \Delta_1, A_1 \supset A_2[I] \Longrightarrow \gamma$ $\supset L$ Rule on previous lines

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma, x{:}s; \Psi, C; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow \forall x{:}s.A[I]} \, \forall R$$

$\Sigma, x{:}s; \Psi \models C$ Weakening Property of $\models$ on $\mathcal{D}$

$\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A[I]$ I.H. on previous line and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \forall x{:}s.A[I]$ $\forall R$ Rule on previous line

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi, C; \Gamma; \Delta_1, [t/x]A[I] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma \vdash t{:}s}}{\Sigma; \Psi, C; \Gamma; \Delta_1, \forall x{:}s.A[I] \Longrightarrow \gamma} \, \forall L$$

$\Sigma; \Psi; \Gamma; \Delta_1, [t/x]A[I] \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta_1, \forall x{:}s.A[I] \Longrightarrow \gamma$ $\forall L$ Rule on previous line and $\mathcal{E}_2$

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A @ I[I']} \, @R$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A @ I[I']$ $@R$ Rule on previous line

Case:

$$\mathcal{E} = \frac{\begin{matrix}\mathcal{E}' \\ \Sigma; \Psi, C; \Gamma; \Delta_1, A[I] \Longrightarrow \gamma\end{matrix}}{\Sigma; \Psi, C; \Gamma; \Delta_1, A \,@\, I[I'] \Longrightarrow \gamma} \ @L$$

$\Sigma; \Psi; \Gamma; \Delta_1, A[I] \Longrightarrow \gamma$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1, A \,@\, I[I'] \Longrightarrow \gamma$ $@L$ Rule on previous line

Case:

$$\mathcal{E} = \frac{\begin{matrix}\mathcal{E}' \\ \Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow A[I]\end{matrix}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \text{affirms}$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I$ affirms Rule on previous line

Case:

$$\mathcal{E} = \frac{\begin{matrix}\mathcal{E}' \\ \Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I\end{matrix}}{\Sigma; \Psi, C; \Gamma; \Delta \Longrightarrow \langle K \rangle A[I]} \ \langle\rangle R$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I$ I.H. on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \langle K \rangle A[I]$ $\langle\rangle R$ Rule on previous line

Case:

$$\mathcal{E} = \frac{\begin{matrix}\mathcal{E}_1 & & \mathcal{E}_2 \\ \Sigma; \Psi, C; \Gamma; \Delta_1, A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I' & \Sigma; \Psi, C \models I \supseteq I'\end{matrix}}{\Sigma; \Psi, C; \Gamma; \Delta_1, \langle K \rangle A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'} \ \langle\rangle L$$

$\Sigma; \Psi; \Gamma; \Delta_1, A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ I.H. on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma; \Psi \models I \supseteq I'$ Cut Property of $\models$ on $\mathcal{D}$ and $\mathcal{E}_2$
$\Sigma; \Psi; \Gamma; \Delta_1, \langle K \rangle A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ $\langle\rangle L$ Rule on previous lines

Case:

$$\mathcal{E} = \frac{\begin{matrix}\mathcal{E}' \\ \Sigma; \Psi, C \models I \supseteq I'\end{matrix}}{\Sigma; \Psi, C; \Gamma; \cdot \Longrightarrow I \supseteq I'[I'']} \ \supseteq R$$

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$$\Sigma; \Psi \models I \supseteq I'$$
$$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow I \supseteq I'[I'']$$

Cut Property of $\models$ on $\mathcal{D}$ and $\mathcal{E}'$
$\supseteq R$ Rule on previous line

Case:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi, C, I \supseteq I'; \Gamma; \Delta_1 \Longrightarrow \gamma}}{\Sigma; \Psi, C; \Gamma; \Delta_1, I \supseteq I'[I''] \Longrightarrow \gamma} \supseteq L$$

$$\Sigma; \Psi, I \supseteq I' \models C$$
$$\Sigma; \Psi, I \supseteq I'; \Gamma; \Delta_1 \Longrightarrow \gamma$$
$$\Sigma; \Psi; \Gamma; \Delta_1, I \supseteq I'[I''] \Longrightarrow \gamma$$

Weakening Property of $\models$ on $\mathcal{D}$
I.H. on previous line and $\mathcal{E}'$
$\supseteq L$ Rule on previous line

$\square$

B.3.2 Substitution Lemma

Lemma 4. If $\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow \gamma$ and $\Sigma \vdash t{:}s$, then $\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x]\gamma$.

Proof. By structural induction on the given derivation.

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma, x{:}s; \Psi \models I \supseteq I'}}{\Sigma, x{:}s; \Psi; \Gamma; P[I] \Longrightarrow P[I']} \; \textsf{init}$$

$$\Sigma; [t/x]\Psi \models [t/x](I \supseteq I')$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](P[I]) \Longrightarrow [t/x](P[I'])$$

Substitution Property of $\models$ on $\mathcal{D}'$
init Rule and definition of substitution
on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma', A[\![I]\!]; \Delta, A[I] \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma', A[\![I]\!]; \Delta \Longrightarrow \gamma} \; \textsf{copy}$$

$$\Sigma; [t/x]\Psi; [t/x](\Gamma', A[\![I]\!]); [t/x](\Delta, A[I]) \Longrightarrow [t/x]\gamma$$
$$\Sigma; [t/x]\Psi; [t/x](\Gamma', A[\![I]\!]); [t/x]\Delta \Longrightarrow [t/x]\gamma$$

I.H. on $\mathcal{D}'$
copy Rule and definition of substitution
on previous line

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1 \Longrightarrow A_1[I]} \quad \overset{\textstyle \mathcal{D}_2}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_2 \Longrightarrow A_2[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2[I]} \; \otimes R$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta_1 \Longrightarrow [t/x](A_1[I])$	I.H. on $\mathcal{D}_1$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta_2 \Longrightarrow [t/x](A_2[I])$	I.H. on $\mathcal{D}_2$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, \Delta_2) \Longrightarrow [t/x](A_1 \otimes A_2[I])$	$\otimes R$ Rule and definition of substitution on previous lines

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1[I], A_2[I] \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1 \otimes A_2[I] \Longrightarrow \gamma} \; \otimes L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1[I], A_2[I]) \Longrightarrow [t/x]\gamma$	I.H. on $\mathcal{D}'$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1 \otimes A_2[I]) \Longrightarrow [t/x]\gamma$	$\otimes L$ Rule and definition of substitution on previous line

Case:

$$\mathcal{D} = \cfrac{}{\Sigma, x{:}s; \Psi; \Gamma; \cdot \Longrightarrow \mathbf{1}[I]} \; \mathbf{1}R$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; \cdot \Longrightarrow [t/x](\mathbf{1}[I])$	$\mathbf{1}R$ Rule and definition of substitution on previous line

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1 \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, \mathbf{1}[I] \Longrightarrow \gamma} \; \mathbf{1}L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta_1 \Longrightarrow [t/x]\gamma$	I.H. on $\mathcal{D}'$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, \mathbf{1}[I]) \Longrightarrow [t/x]\gamma$	$\mathbf{1}L$ Rule and definition of substitution on previous line

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1[I]} \quad \overset{\textstyle \mathcal{D}_2}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_2[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1 \mathbin{\&} A_2[I]} \; \mathbin{\&} R$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1[I]) \qquad\qquad \text{I.H. on } \mathcal{D}_1$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_2[I]) \qquad\qquad \text{I.H. on } \mathcal{D}_2$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1 \,\&\, A_2[I]) \qquad \&R \text{ Rule and definition of substitution}$$
$$\text{on previous lines}$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1[I] \Longrightarrow \gamma\end{array}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1 \,\&\, A_2[I] \Longrightarrow \gamma} \; \&L_1$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1 \,\&\, A_2[I]) \Longrightarrow [t/x]\gamma \qquad \&L_1 \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_2[I] \Longrightarrow \gamma\end{array}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1 \,\&\, A_2[I] \Longrightarrow \gamma} \; \&L_2$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_2[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1 \,\&\, A_2[I]) \Longrightarrow [t/x]\gamma \qquad \&L_2 \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \dfrac{}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow \top[I]} \; \top R$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](\top[I]) \qquad \top R \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1[I]\end{array}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I]} \; \oplus R_1$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1[I]) \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1 \oplus A_2[I]) \qquad \oplus R_1 \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_2[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_2$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_2[I])$	I.H. on $\mathcal{D}'$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1 \oplus A_2[I])$	$\oplus R_2$ Rule and definition of substitution on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1[I] \Longrightarrow \gamma} \quad \overset{\mathcal{D}_2}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_2[I] \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1 \oplus A_2[I] \Longrightarrow \gamma} \ \oplus L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1[I]) \Longrightarrow [t/x]\gamma$	I.H. on $\mathcal{D}_1$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_2[I]) \Longrightarrow [t/x]\gamma$	I.H. on $\mathcal{D}_2$
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1 \oplus A_2[I]) \Longrightarrow [t/x]\gamma$	$\oplus L$ Rule and definition of substitution on previous lines

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma, x{:}s, i{:}\mathsf{interval}; \Psi, I \supseteq i; \Gamma; \Delta, A_1[i] \Longrightarrow A_2[i]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1 \multimap A_2[I]} \ \multimap R$$

$\Sigma, i{:}\mathsf{interval}; [t/x](\Psi, I \supseteq i); [t/x]\Gamma; [t/x](\Delta, A_1[i]) \Longrightarrow [t/x](A_2[i])$	I.H. on previous line and $\mathcal{D}'$
$\Sigma, i{:}\mathsf{interval}; [t/x]\Psi, [t/x]I \supseteq i; [t/x]\Gamma; [t/x]\Delta, [t/x]A_1[i] \Longrightarrow [t/x]A_2[i]$	Definition of substitution and i is fresh
$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1 \multimap A_2[I])$	$\multimap R$ Rule and definition of substitution on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1 \Longrightarrow A_1[I']} \quad \overset{\mathcal{D}_2}{\Sigma, x{:}s; \Psi \models I \supseteq I'} \quad \overset{\mathcal{D}_3}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_2, A_2[I'] \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2[I] \Longrightarrow \gamma} \ \multimap L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta_1 \Longrightarrow [t/x](A_1[I']) \qquad\qquad$ I.H. on $\mathcal{D}_1$

$\Sigma; [t/x]\Psi; [t/x]\Gamma \models [t/x](I \supseteq I') \qquad\qquad$ Substitution Property of $\models$ on $\mathcal{D}_2$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_2, A_2[I']) \Longrightarrow [t/x]\gamma \qquad\qquad$ I.H. on $\mathcal{D}_3$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, \Delta_2, A_1 \multimap A_2[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \multimap L$ Rule and definition of
substitution on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \cdot \Longrightarrow A[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \cdot \Longrightarrow {!}A[I]} \ {!}R$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; \cdot \Longrightarrow [t/x](A[I]) \qquad\qquad$ I.H. on $\mathcal{D}'$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; \cdot \Longrightarrow [t/x]({!}A[I]) \qquad\qquad {!}R$ Rule and definition of substitution
on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma, A[\![I]\!]; \Delta_1 \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, {!}A[I] \Longrightarrow \gamma} \ {!}L$$

$\Sigma; [t/x]\Psi; [t/x](\Gamma, A[\![I]\!]); [t/x]\Delta_1 \Longrightarrow [t/x]\gamma \qquad\qquad$ I.H. on $\mathcal{D}'$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, {!}A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad {!}L$ Rule and definition of substitution
on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma, x{:}s, i{:}\mathsf{interval}; \Psi, I \supseteq i; \Gamma, A_1[\![i]\!]; \Delta \Longrightarrow A_2[i]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I]} \ \supset R$$

$\Sigma, i{:}\mathsf{interval}; [t/x](\Psi, I \supseteq i); [t/x](\Gamma, A_1[\![i]\!]); [t/x]\Delta \Longrightarrow [t/x](A_2[i]) \qquad\qquad$ I.H. on $\mathcal{D}'$

$\Sigma, i{:}\mathsf{interval}; [t/x]\Psi, [t/x]I \supseteq i; [t/x]\Gamma, [t/x]A_1[\![i]\!]; [t/x]\Delta \Longrightarrow [t/x]A_2[i]$

$\qquad\qquad$ Definition of substitution and i is fresh

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A_1 \supset A_2[I]) \qquad\qquad \supset R$ Rule and
definition of substitution on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \cdot \Longrightarrow A_1[I']} \quad \overset{\mathcal{D}_2}{\Sigma, x{:}s; \Psi \models I \supseteq I'} \quad \overset{\mathcal{D}_3}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_2[I'] \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A_1 \supset A_2[I] \Longrightarrow \gamma} \ \supset L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; \cdot \Longrightarrow [t/x](A_1[I']) \qquad\qquad$ I.H. on $\mathcal{D}_1$

$\Sigma; [t/x]\Psi \models [t/x](I \supseteq I') \qquad\qquad$ Substitution Property of $\models$ on $\mathcal{D}_2$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_2[I']) \Longrightarrow [t/x]\gamma \qquad\qquad$ I.H. on $\mathcal{D}_3$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A_1 \supset A_2[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \supset L$ Rule and definition of substitution
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s, x'{:}s'; \Psi; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow \forall x'{:}s'.A[I]} \ \forall R$$

$\Sigma, x'{:}s'; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A[I]) \qquad\qquad$ I.H. on $\mathcal{D}'$ and x' is fresh

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](\forall x'{:}s'.A[I]) \qquad\qquad \forall R$ Rule and definition of substitution
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ on previous line

Case: There are two cases for the $\forall L$ rule. The substituted term may be x or it may not be.

Subcase:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, [x/x']A[I] \Longrightarrow \gamma} \quad \overset{\textstyle \mathcal{D}_2}{\Sigma, x{:}s \vdash x{:}s}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, \forall x'{:}s.A[I] \Longrightarrow \gamma} \ \forall L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, [x/x']A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad$ I.H. on $\mathcal{D}_1$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, [t/x']A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad$ Definition of substitution
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ on previous line

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, \forall x'{:}s.A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \forall L$ Rule on previous line

Subcase:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, [t'/x']A[I] \Longrightarrow \gamma} \quad \overset{\textstyle \mathcal{D}_2}{\Sigma, x{:}s \vdash t'{:}s'}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, \forall x'{:}s'.A[I] \Longrightarrow \gamma} \ \forall L$$

$\Sigma \vdash t'{:}s'$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, [t'/x']A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad$ I.H. on $\mathcal{D}_1$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, \forall x'{:}s'.A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \forall L$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A @ I[I']} \ @R$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A[I]) \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A @ I[I']) \qquad\qquad @R \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A[I] \Longrightarrow \gamma}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, A @ I[I'] \Longrightarrow \gamma} \ @L$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A[I]) \Longrightarrow [t/x]\gamma \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, A @ I[I']) \Longrightarrow [t/x]\gamma \qquad\qquad @L \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \text{affirms}$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](A[I]) \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x]((K \text{ affirms } A) \text{ at } I) \qquad\qquad \text{affirms Rule and definition of}$$
$$\text{substitution on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow \langle K \rangle A[I]} \ \langle\rangle R$$

$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x]((K \text{ affirms } A) \text{ at } I) \qquad\qquad \text{I.H. on } \mathcal{D}'$$
$$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x]\Delta \Longrightarrow [t/x](\langle K \rangle A[I]) \qquad\qquad \langle\rangle R \text{ Rule and definition of substitution}$$
$$\text{on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\textstyle \mathcal{D}_1}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, B[I] \Longrightarrow (K \text{ affirms } A) \text{ at } I'} \quad \overset{\textstyle \mathcal{D}_2}{\Sigma, x{:}s; \Psi \models I \supseteq I'}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, \langle K \rangle B[I] \Longrightarrow (K \text{ affirms } A) \text{ at } I'} \ \langle\rangle L$$

$\Sigma; [t/x]\Psi; [t/x]\Gamma, ; [t/x](\Delta_1, B[I]) \Longrightarrow [t/x]((K \text{ affirms } A) \text{ at } I')$ I.H. on $\mathcal{D}_1$

$\Sigma; [t/x]\Psi \models [t/x](I \supseteq I')$ Substitution Property of $\models$ on $\mathcal{D}_2$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, \langle K \rangle B[I]) \Longrightarrow [t/x]((K \text{ affirms } A) \text{ at } I')$ $\langle \rangle L$ Rule and
definition of substitution on previous lines

Case:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma, x{:}s; \Psi \models I \supseteq I' \end{array}}{\Sigma, x{:}s; \Psi; \Gamma; \cdot \Longrightarrow I \supseteq I'[I'']} \supseteq R$$

$\Sigma; [t/x]\Psi \models [t/x](I \supseteq I')$ Substitution Property of $\models$ on $\mathcal{D}'$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; \cdot \Longrightarrow [t/x](I \supseteq I'[I''])$ $\supseteq R$ Rule and definition of substitution
on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma, x{:}s; \Psi, I \supseteq I'; \Gamma; \Delta_1 \Longrightarrow \gamma \end{array}}{\Sigma, x{:}s; \Psi; \Gamma; \Delta_1, I \supseteq I'[I''] \Longrightarrow \gamma} \supseteq L$$

$\Sigma; [t/x](\Psi, I \supseteq I'); [t/x]\Gamma; [t/x]\Delta_1 \Longrightarrow [t/x]\gamma$ I.H. on $\mathcal{D}'$

$\Sigma; [t/x]\Psi; [t/x]\Gamma; [t/x](\Delta_1, I \supseteq I'[I'']) \Longrightarrow [t/x]\gamma$ $\supseteq L$ Rule and definition of substitution
on previous line

$\square$

B.3.3   Admissibility of Cut Proof

Theorem 1.

1. If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]$ and $\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow \gamma$, then $\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \gamma$.
2. If $\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A[I]$ and $\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow \gamma$, then $\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow \gamma$.
3. If $\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I$ and $\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ and $\Sigma; \Psi \models I \supseteq I'$, then $\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow (K \text{ affirms } B) \text{ at } I'$.

Proof. By simultaneous induction. Part 1 is proven by nested induction on the size of the cut formula, A, and on the size of the given derivations. Part 2 is proven by structural induction on the second given derivation, where we may appeal to part 1 even on larger derivations. Part 3 is proven by structural induction on the first given derivation.

Part 1:

Case: Initial Cut

     Subcase:

$$\mathcal{E} = \cfrac{\cfrac{\mathcal{E}'}{\Sigma; \Psi \models I \supseteq I' \quad (P \text{ atomic})}}{\Sigma; \Psi; \Gamma; P[I] \Longrightarrow P[I']} \text{ init}$$

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow P[I']$ Theorem 3 on $\mathcal{D}$ and $\mathcal{E}'$

Case: Principal Cuts

    Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow A_1[I]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Gamma; \Delta_2 \Longrightarrow A_2[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2[I]} \otimes R$$

$$\text{and}$$

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta', A_1[I], A_2[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta', A_1 \otimes A_2[I] \Longrightarrow \gamma} \otimes L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, A_2[I] \Longrightarrow \gamma$ I.H.(1) on A_1, $\mathcal{D}_1$, and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, \Delta_2 \Longrightarrow \gamma$ I.H.(1) on A_2, $\mathcal{D}_2$, and previous line

    Subcase:

$$\mathcal{D} = \cfrac{}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow \mathbf{1}[I]} \mathbf{1}R$$

$$\text{and}$$

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta', \mathbf{1}[I] \Longrightarrow \gamma} \mathbf{1}L$$

$\Sigma; \Psi; \Gamma; \Delta', \cdot \Longrightarrow \gamma$ $\mathcal{E}'$

    Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1[I]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \mathbin{\&} A_2[I]} \mathbin{\&}R$$

$$\text{and}$$

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta', A_1[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta', A_1 \mathbin{\&} A_2[I] \Longrightarrow \gamma} \mathbin{\&}L_1$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \gamma$ I.H.(1) on A_1, $\mathcal{D}_1$, and $\mathcal{E}'$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1[I]} \quad \overset{\mathcal{D}_2}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_2[I]}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1 \mathbin{\&} A_2[I]} \ \&R$$

and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma;\Psi;\Gamma;\Delta',A_2[I] \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma;\Delta',A_1 \mathbin{\&} A_2[I] \Longrightarrow \gamma} \ \&L_2$$

$$\Sigma;\Psi;\Gamma;\Delta',\Delta \Longrightarrow \gamma \hspace{4cm} \text{I.H.(1) on } A_2,\ \mathcal{D}_2,\ \text{and } \mathcal{E}'$$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1[I]}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_1$$

and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma;\Psi;\Gamma;\Delta',A_1[I] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma;\Psi;\Gamma;\Delta',A_2[I] \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma;\Delta',A_1 \oplus A_2[I] \Longrightarrow \gamma} \ \oplus L$$

$$\Sigma;\Psi;\Gamma;\Delta',\Delta \Longrightarrow \gamma \hspace{4cm} \text{I.H.(1) on } A_1,\ \mathcal{D}',\ \text{and } \mathcal{E}_1$$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_2[I]}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1 \oplus A_2[I]} \ \oplus R_2$$

and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma;\Psi;\Gamma;\Delta',A_1[I] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma;\Psi;\Gamma;\Delta',A_2[I] \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma;\Delta',A_1 \oplus A_2[I] \Longrightarrow \gamma} \ \oplus L$$

$$\Sigma;\Psi;\Gamma;\Delta',\Delta \Longrightarrow \gamma \hspace{4cm} \text{I.H.(1) on } A_2,\ \mathcal{D}',\ \text{and } \mathcal{E}_2$$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma, i\text{:interval};\Psi, I \supseteq i;\Gamma;\Delta, A_1[i] \Longrightarrow A_2[i]}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1 \multimap A_2[I]} \ \multimap R$$

and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta_1' \Longrightarrow A_1[I']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi \models I \supseteq I'} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi; \Gamma; \Delta_2', A_2[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', \Delta_2', A_1 \multimap A_2[I] \Longrightarrow \gamma} \ \multimap L$$

$\Sigma; [I'/i](\Psi, I \supseteq i); [I'/i]\Gamma; [I'/i](\Delta, A_1[i]) \Longrightarrow [I'/i](A_2[i])$ Lemma 4 on $\mathcal{D}'$

$\Sigma; \Psi, I \supseteq I'; \Gamma; \Delta, A_1[I'] \Longrightarrow A_2[I']$ Definition of Substitution and i is fresh

       on previous line

$\Sigma; \Psi; \Gamma; \Delta, A_1[I'] \Longrightarrow A_2[I']$ Lemma 3 on $\mathcal{E}_2$ and previous line

$\Sigma; \Psi; \Gamma; \Delta, \Delta_1' \Longrightarrow A_2[I']$ I.H.(1) on A_1, $\mathcal{E}_1$, and previous line

$\Sigma; \Psi; \Gamma; \Delta_2', \Delta, \Delta_1' \Longrightarrow \gamma$ I.H.(1) on A_2, previous line, and $\mathcal{E}_3$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A_1[I]}}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow \ !A_1[I]} \ !R$$

and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A_1[\![I]\!]; \Delta' \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta', !A_1[I] \Longrightarrow \gamma} \ !L$$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow \gamma$ I.H.(2) on A_1, $\mathcal{D}'$, and $\mathcal{E}'$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma, i:\mathsf{interval}; \Psi, I \supseteq i; \Gamma, A_1[\![i]\!]; \Delta \Longrightarrow A_2[i]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_1 \supset A_2[I]} \ \supset R$$

and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow A_1[I']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi \models I \supseteq I'} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi; \Gamma; \Delta', A_2[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta', A_1 \supset A_2[I] \Longrightarrow \gamma} \ \supset L$$

$\Sigma; [I'/i](\Psi, I \supseteq i); [I'/i](\Gamma, A_1[\![i]\!]); [I'/i]\Delta \Longrightarrow [I'/i](A_2[i])$ Lemma 4 on $\mathcal{D}'$

$\Sigma; \Psi, I \supseteq I'; \Gamma, A_1[\![I']\!]; \Delta \Longrightarrow A_2[I']$

       Definition of Substitution and i is fresh on previous line

$\Sigma; \Psi; \Gamma, A_1[\![I']\!]; \Delta \Longrightarrow A_2[I']$ Lemma 3 on $\mathcal{E}_2$ and previous line

$\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A_2[I']$ I.H.(2) on A_1, $\mathcal{E}_1$, and previous line

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \gamma$ I.H.(1) on A_2, previous line, and $\mathcal{E}_3$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma, x:s; \Psi; \Gamma; \Delta \Longrightarrow A_1[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow \forall x{:}s.A_1[I]} \ \forall R$$

and

$$\mathcal{E} = \frac{\begin{array}{cc} \mathcal{E}_1 & \mathcal{E}_2 \\ \Sigma;\Psi;\Gamma;\Delta',[t/x]A_1[I] \Longrightarrow \gamma & \Sigma \models t{:}s \end{array}}{\Sigma;\Psi;\Gamma;\Delta',\forall x{:}s.A_1[I] \Longrightarrow \gamma} \ \forall L$$

$\Sigma;\Psi;\Gamma;\Delta \Longrightarrow [t/x]A_1[I]$ Lemma 4 on $\mathcal{E}_2$ and $\mathcal{D}'$

$\Sigma;\Psi;\Gamma;\Delta',\Delta \Longrightarrow \gamma$ I.H.(1) on $[t/x]A_1$, previous line, and $\mathcal{E}_1$

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1[I'] \end{array}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow A_1 @ I'[I]} \ @R$$

and

$$\mathcal{E} = \frac{\begin{array}{c} \mathcal{E}' \\ \Sigma;\Psi;\Gamma;\Delta',A_1[I'] \Longrightarrow \gamma \end{array}}{\Sigma;\Psi;\Gamma;\Delta',A_1 @ I'[I] \Longrightarrow \gamma} \ @L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta \Longrightarrow \gamma$ I.H.(1) on A_1, $\mathcal{D}'$, and $\mathcal{E}'$

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma;\Psi;\Gamma;\Delta \Longrightarrow (K \text{ affirms } A_1) \text{ at } I \end{array}}{\Sigma;\Psi;\Gamma;\Delta \Longrightarrow \langle K \rangle A_1[I]} \ \langle\rangle R$$

and

$$\mathcal{E} = \frac{\begin{array}{cc} \mathcal{E}_1 & \mathcal{E}_2 \\ \Sigma;\Psi;\Gamma;\Delta',A_1[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I' & \Sigma;\Psi \models I \supseteq I' \end{array}}{\Sigma;\Psi;\Gamma;\Delta',\langle K \rangle A_1[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'} \ \langle\rangle L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ I.H.(3) on A_1, $\mathcal{D}'$, $\mathcal{E}_1$, and $\mathcal{E}_2$

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma;\Psi \models I' \supseteq I'' \end{array}}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow I' \in I''[I]} \ \supseteq R$$

and

$$\mathcal{E} = \frac{\begin{array}{c} \mathcal{E}' \\ \Sigma;\Psi,I' \supseteq I'';\Gamma;\Delta' \Longrightarrow \gamma \end{array}}{\Sigma;\Psi;\Gamma;\Delta',I' \supseteq I''[I] \Longrightarrow \gamma} \ \supseteq L$$

$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow \gamma$ Lemma 3 on $\mathcal{D}'$ and $\mathcal{E}'$

Case: Left Commutative Cuts

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma',B[\![I']\!];\Delta,B[I'] \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma',B[\![I']\!];\Delta \Longrightarrow A[I]} \text{ copy}$$

$\Sigma;\Psi;\Gamma',B[\![I']\!];\Delta',\Delta,B[I'] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$
$\Sigma;\Psi;\Gamma',B[\![I']\!];\Delta',\Delta \Longrightarrow \gamma$ copy Rule on previous line

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma;\Delta_1,B_1[I'],B_2[I'] \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma;\Delta_1,B_1 \otimes B_2[I'] \Longrightarrow A[I]} \otimes L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,B_1[I'],B_2[I'] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,B_1 \otimes B_2[I'] \Longrightarrow \gamma$ $\otimes L$ Rule on previous line

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma;\Delta_1,\mathbf{1}[I'] \Longrightarrow A[I]} \mathbf{1}L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1 \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,\mathbf{1}[I'] \Longrightarrow \gamma$ $\mathbf{1}L$ Rule on previous line

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma;\Delta_1,B_1[I'] \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma;\Delta_1,B_1 \mathbin{\&} B_2[I'] \Longrightarrow A[I]} \mathbin{\&}L_1$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,B_1[I'] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,B_1 \mathbin{\&} B_2[I'] \Longrightarrow \gamma$ $\mathbin{\&}L_1$ Rule on previous line

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\\Sigma;\Psi;\Gamma;\Delta_1,B_2[I'] \Longrightarrow A[I]\end{array}}{\Sigma;\Psi;\Gamma;\Delta_1,B_1 \mathbin{\&} B_2[I'] \Longrightarrow A[I]} \mathbin{\&}L_2$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,B_2[I'] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1,B_1 \mathbin{\&} B_2[I'] \Longrightarrow \gamma$ $\mathbin{\&}L_2$ Rule on previous line

Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1, B_1[I'] \Longrightarrow A[I]} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I'] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \oplus B_2[I'] \Longrightarrow A[I]} \; \oplus L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B_1[I'] \Longrightarrow \gamma$	I.H.(1) on A, $\mathcal{D}_1$, and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B_2[I'] \Longrightarrow \gamma$	I.H.(1) on A, $\mathcal{D}_2$, and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B_1 \oplus B_2[I'] \Longrightarrow \gamma$	$\oplus L$ Rule on previous lines

Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow B_1[I'']} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I' \sqsupseteq I''} \quad \overset{\mathcal{D}_3}{\Sigma; \Psi; \Gamma; \Delta_2, B_2[I''] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2[I'] \Longrightarrow A[I]} \; \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_2, B_2[I''] \Longrightarrow \gamma$	I.H.(1) on A, $\mathcal{D}_3$, and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta_1, \Delta', \Delta_2, B_1 \multimap B_2[I'] \Longrightarrow \gamma$	$\multimap L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma, B[\![I']\!]; \Delta_1 \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, !B[I'] \Longrightarrow A[I]} \; !L$$

$\Sigma; \Psi; \Gamma, B[\![I']\!]; \Delta', A[I] \Longrightarrow \gamma$	Weakening on $\mathcal{E}$
$\Sigma; \Psi; \Gamma, B[\![I']\!]; \Delta', \Delta_1 \Longrightarrow \gamma$	I.H.(1) on A, $\mathcal{D}'$, and previous line
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, !B[I'] \Longrightarrow \gamma$	$!L$ Rule on previous line

Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow B_1[I'']} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I' \sqsupseteq I''} \quad \overset{\mathcal{D}_3}{\Sigma; \Psi; \Gamma; \Delta_1, B_2[I''] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, B_1 \supset B_2[I'] \Longrightarrow A[I]} \; \supset L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B_2[I''] \Longrightarrow \gamma$	I.H.(1) on A, $\mathcal{D}_3$, and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B_1 \supset B_2[I'] \Longrightarrow \gamma$	$\supset L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1, [t/x]B[I'] \Longrightarrow A[I]} \quad \overset{\mathcal{D}_2}{\Sigma \models t{:}s}}{\Sigma; \Psi; \Gamma; \Delta_1, \forall x{:}s.B[I'] \Longrightarrow A[I]} \; \forall L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, [t/x]B[I'] \Longrightarrow \gamma$	I.H.(1) on A, $\mathcal{D}_1$, and $\mathcal{E}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, \forall x{:}s.B[I'] \Longrightarrow \gamma$	$\forall L$ Rule on previous line and $\mathcal{D}_2$

Subcase:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta_1, B[I'] \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, B @ I'[I''] \Longrightarrow A[I]} \ @L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B[I'] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, B @ I'[I''] \Longrightarrow \gamma$ $@L$ Rule on previous line

Subcase:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1 \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta_1, I' \supseteq I''[I'''] \Longrightarrow A[I]} \ \supseteq L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta', A[I] \Longrightarrow \gamma$ Weakening on $\mathcal{E}$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta', \Delta_1 \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}'$, and previous line

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, I' \supseteq I''[I'''] \Longrightarrow \gamma$ $\supseteq L$ Rule on previous line

Case: Right Commutative Cuts

Subcase:

$$\mathcal{E} = \cfrac{\cfrac{\mathcal{E}'}{\Sigma; \Psi; \Gamma', B[\![I']\!]; \Delta', A[I], B[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma', B[\![I']\!]; \Delta', A[I] \Longrightarrow \gamma} \ \mathsf{copy}$$

$\Sigma; \Psi; \Gamma', B[\![I']\!]; \Delta', \Delta, B[I'] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma', B[\![I']\!]; \Delta', \Delta \Longrightarrow \gamma$ copy Rule on previous line

Subcase: The last inference of $\mathcal{E}$ is the $\otimes R$ rule. There are two subsubcases; the resource $A[I]$ may be sent to the derivation of the left premise, or it may be sent to the derivation of the right premise.

Subsubcase:

$$\mathcal{E} = \cfrac{\cfrac{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta'_1, A[I] \Longrightarrow B_1[I']} \quad \cfrac{\mathcal{E}_2}{\Sigma; \Psi; \Gamma; \Delta'_2 \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma; \Delta'_1, \Delta'_2, A[I] \Longrightarrow B_1 \otimes B_2[I']} \ \otimes R$$

$\Sigma; \Psi; \Gamma; \Delta'_1, \Delta \Longrightarrow B_1[I']$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta'_1, \Delta, \Delta'_2 \Longrightarrow B_1 \otimes B_2[I']$ $\otimes R$ Rule on previous line and $\mathcal{E}_2$

Subsubcase:

$$\mathcal{E} = \cfrac{\cfrac{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta'_1 \Longrightarrow B_1[I']} \quad \cfrac{\mathcal{E}_2}{\Sigma; \Psi; \Gamma; \Delta'_2, A[I] \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma; \Delta'_1, \Delta'_2, A[I] \Longrightarrow B_1 \otimes B_2[I']} \ \otimes R$$

$$\Sigma; \Psi; \Gamma; \Delta_2', \Delta \Longrightarrow B_2[I'] \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}_2$$
$$\Sigma; \Psi; \Gamma; \Delta_1', \Delta_2', \Delta \Longrightarrow B_1 \otimes B_2[I'] \qquad\qquad \otimes R \text{ Rule on } \mathcal{E}_1 \text{ and previous line}$$

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta_1', B_1[I'], B_2[I'], A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', B_1 \otimes B_2[I'], A[I] \Longrightarrow \gamma} \ \otimes L$$

$$\Sigma; \Psi; \Gamma; \Delta_1', B_1[I'], B_2[I'], \Delta \Longrightarrow \gamma \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta_1', B_1 \otimes B_2[I'], \Delta \Longrightarrow \gamma \qquad\qquad \otimes L \text{ Rule on previous line}$$

Note: There is no case here for the $1R$ rule. This rule requires the linear context in the conclusion to be empty. But, the derivation $\mathcal{E}$ must have the cut formula $A[I]$ as an assumption in the linear context. So, it is impossible for $\mathcal{E}$ to end with the $1R$ rule.

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta_1', A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', \mathbf{1}[I'], A[I] \Longrightarrow \gamma} \ \mathbf{1}L$$

$$\Sigma; \Psi; \Gamma; \Delta_1', \Delta \Longrightarrow \gamma \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta_1', \mathbf{1}[I'], \Delta \Longrightarrow \gamma \qquad\qquad \mathbf{1}L \text{ Rule on previous line}$$

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1[I']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1 \,\&\, B_2[I']} \ \&R$$

$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1[I'] \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}_1$$
$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_2[I'] \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}_2$$
$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1 \,\&\, B_2[I'] \qquad\qquad \&R \text{ Rule on previous lines}$$

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta_1', B_1[I'], A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', B_1 \,\&\, B_2[I'], A[I] \Longrightarrow \gamma} \ \&L_1$$

$$\Sigma; \Psi; \Gamma; \Delta_1', B_1[I'], \Delta \Longrightarrow \gamma \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta_1', B_1 \,\&\, B_2[I'], \Delta \Longrightarrow \gamma \qquad\qquad \&L_1 \text{ Rule on previous line}$$

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta_1', B_2[I'], A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', B_1 \,\&\, B_2[I'], A[I] \Longrightarrow \gamma} \ \&L_2$$

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$\Sigma; \Psi; \Gamma; \Delta'_1, B_2[I'], \Delta \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta'_1, B_1 \mathbin{\&} B_2[I'], \Delta \Longrightarrow \gamma$ $\&L_2$ Rule on previous line

Subcase:

$$\mathcal{E} = \dfrac{}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow \top[I']} \ \top R$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \top[I']$ $\top R$ Rule

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1[I']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1 \oplus B_2[I']} \ \oplus R_1$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1[I']$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1 \oplus B_2[I']$ $\oplus R_1$ Rule on previous line

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1 \oplus B_2[I']} \ \oplus R_2$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_2[I']$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1 \oplus B_2[I']$ $\oplus R_2$ Rule on previous line

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta'_1, B_1[I'], A[I] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi; \Gamma; \Delta'_1, B_2[I'], A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta'_1, B_1 \oplus B_2[I'], A[I] \Longrightarrow \gamma} \ \oplus L$$

$\Sigma; \Psi; \Gamma; \Delta'_1, B_1[I'], \Delta \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta'_1, B_2[I'], \Delta \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}_2$

$\Sigma; \Psi; \Gamma; \Delta'_1, B_1 \oplus B_2[I'], \Delta \Longrightarrow \gamma$ $\oplus L$ Rule on previous lines

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; \Delta', A[I], B_1[i'] \Longrightarrow B_2[i']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1 \multimap B_2[I']} \ \multimap R$$

$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; \Delta \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; \Delta', \Delta, B_1[i'] \Longrightarrow B_2[i']$ I.H.(1) on A, previous line, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1 \multimap B_2[I']$ $\multimap R$ Rule on previous line

Subcase: The last inference of $\mathcal{E}$ is the $\multimap L$ rule. There are two subsubcases; the resource $A[I]$ may be sent to the derivation of the left premise, or it may be sent to the derivation of the right premise.

Subsubcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta_1', A[I] \Longrightarrow B_1[I'']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi \models I' \supseteq I''} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi; \Gamma; \Delta_2', B_2[I''] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', \Delta_2', B_1 \multimap B_2[I'], A[I] \Longrightarrow \gamma} \; \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta_1', \Delta \Longrightarrow B_1[I'']$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta_1', \Delta, \Delta_2', B_1 \multimap B_2[I'] \Longrightarrow \gamma$ $\multimap L$ Rule on previous line, $\mathcal{E}_2$, and $\mathcal{E}_3$

Subsubcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta_1' \Longrightarrow B_1[I'']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi \models I' \supseteq I''} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi; \Gamma; \Delta_2', A[I], B_2[I''] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', \Delta_2', B_1 \multimap B_2[I'], A[I] \Longrightarrow \gamma} \; \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta_2', \Delta, B_2[I''] \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}_3$

$\Sigma; \Psi; \Gamma; \Delta_1', \Delta_2', B_1 \multimap B_2[I'], \Delta \Longrightarrow \gamma$ $\multimap L$ Rule on $\mathcal{E}_1$, $\mathcal{E}_2$, and previous line

Note: There is no case here for the $!R$ rule. This rule requires the linear context in the conclusion to be empty. But, the derivation $\mathcal{E}$ must have the cut formula $A[I]$ as an assumption in the linear context. So, it is impossible for $\mathcal{E}$ to end with the $!R$ rule.

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, B[\![I']\!]; \Delta_1', A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', !B[I'], A[I] \Longrightarrow \gamma} \; !L$$

$\Sigma; \Psi; \Gamma, B[\![I']\!]; \Delta \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma; \Psi; \Gamma, B[\![I']\!]; \Delta_1', \Delta \Longrightarrow \gamma$ I.H.(1) on A, previous line, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta_1', !B[I'], \Delta \Longrightarrow \gamma$ $!L$ rule on previous line

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma, i':\text{interval}; \Psi, I' \supseteq i'; \Gamma, B_1[\![i']\!]; \Delta', A[I] \Longrightarrow B_2[i']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B_1 \supset B_2[I']} \; \supset R$$

$\Sigma, i':\text{interval}; \Psi, I' \supseteq i'; \Gamma, B_1[\![i']\!]; \Delta \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma, i':\text{interval}; \Psi, I' \supseteq i'; \Gamma, B_1[\![i']\!]; \Delta', \Delta \Longrightarrow B_2[i']$ I.H.(1) on A, previous line, and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B_1 \supset B_2[I']$ $\supset R$ Rule on previous line

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \cdot \Longrightarrow B_1[I'']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi \models I' \supseteq I''} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi; \Gamma; \Delta_1', A[I], B_2[I''] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta_1', B_1 \supset B_2[I'], A[I] \Longrightarrow \gamma} \; \supset L$$

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$$\Sigma; \Psi; \Gamma; \Delta'_1, \Delta, B_2[I''] \Longrightarrow \gamma \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}_3$$
$$\Sigma; \Psi; \Gamma; \Delta'_1, B_1 \supset B_2[I'], \Delta \Longrightarrow \gamma \qquad\qquad \supset L \text{ Rule on } \mathcal{E}_1, \mathcal{E}_2 \text{ and previous line}$$

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma, x{:}s; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B[I']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow \forall x{:}s.B[I']} \ \forall R$$

$$\Sigma, x{:}s; \Psi; \Gamma; \Delta \Longrightarrow A[I] \qquad\qquad \text{Weakening on } \mathcal{D}$$
$$\Sigma, x{:}s; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B[I'] \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \forall x{:}s.B[I'] \qquad\qquad \forall R \text{ Rule on previous line}$$

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma; \Delta'_1, [t/x]B[I'], A[I] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma \models t{:}s}}{\Sigma; \Psi; \Gamma; \Delta'_1, \forall x{:}s.B[I'], A[I] \Longrightarrow \gamma} \ \forall L$$

$$\Sigma; \Psi; \Gamma; \Delta'_1, [t/x]B[I'], \Delta \Longrightarrow \gamma \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}_1$$
$$\Sigma; \Psi; \Gamma; \Delta'_1, \forall x{:}s.B[I'], \Delta \Longrightarrow \gamma \qquad\qquad \forall L \text{ Rule on previous line and } \mathcal{E}_2$$

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B[I']}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow B @ I'[I'']} \ @R$$

$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B[I'] \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow B @ I'[I''] \qquad\qquad @R \text{ Rule on previous line}$$

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta'_1, B[I'], A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma; \Delta'_1, B @ I'[I''], A[I] \Longrightarrow \gamma} \ @L$$

$$\Sigma; \Psi; \Gamma; \Delta'_1, B[I'], \Delta \Longrightarrow \gamma \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta'_1, B @ I'[I''], \Delta \Longrightarrow \gamma \qquad\qquad @L \text{ Rule on previous line}$$

Subcase:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'}}{\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow \langle K \rangle B[I']} \ \langle\rangle R$$

$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow (K \text{ affirms } B) \text{ at } I' \qquad\qquad \text{I.H.(1) on } A, \mathcal{D}, \text{ and } \mathcal{E}'$$
$$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow \langle K \rangle B[I'] \qquad\qquad \langle\rangle R \text{ Rule on previous line}$$

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle\mathcal{E}_1}{\Sigma;\Psi;\Gamma;\Delta_1', B[I'], A[I] \Longrightarrow (K \text{ affirms } D) \text{ at } I'' \quad \overset{\displaystyle\mathcal{E}_2}{\Sigma;\Psi \models I' \supseteq I''}}{\Sigma;\Psi;\Gamma;\Delta_1', \langle K\rangle B[I'], A[I] \Longrightarrow (K \text{ affirms } D) \text{ at } I''}} \; \langle\rangle L$$

$\Sigma;\Psi;\Gamma;\Delta_1', B[I'], \Delta \Longrightarrow (K \text{ affirms } D) \text{ at } I''$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}_1$

$\Sigma;\Psi;\Gamma;\Delta_1', \langle K\rangle B[I'], \Delta \Longrightarrow (K \text{ affirms } D) \text{ at } I''$ $\langle\rangle L$ Rule on previous line and $\mathcal{E}_2$

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle\mathcal{E}'}{\Sigma;\Psi;\Gamma;\Delta', A[I] \Longrightarrow B[I']}}{\Sigma;\Psi;\Gamma;\Delta', A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'} \; \text{affirms}$$

$\Sigma;\Psi;\Gamma;\Delta', \Delta \Longrightarrow B[I']$ I.H.(1) on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Sigma;\Psi;\Gamma;\Delta', \Delta \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ affirms Rule on previous line

Note: There is no case here for the $\supseteq R$ rule. This rule requires the linear context in the conclusion to be empty. But, the derivation $\mathcal{E}$ must have the cut formula $A[I]$ as an assumption in the linear context. So, it is impossible for $\mathcal{E}$ to end with the $\supseteq R$ rule.

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle\mathcal{E}'}{\Sigma;\Psi, I' \supseteq I''; \Gamma; \Delta_1', A[I] \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma;\Delta_1', I' \supseteq I''[I'''], A[I] \Longrightarrow \gamma} \; \supseteq L$$

$\Sigma;\Psi, I' \supseteq I''; \Gamma; \Delta \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma;\Psi, I' \supseteq I''; \Gamma; \Delta_1', \Delta \Longrightarrow \gamma$ I.H.(1) on A, previous line, and $\mathcal{E}'$

$\Sigma;\Psi;\Gamma;\Delta_1', I' \supseteq I''[I'''], \Delta \Longrightarrow \gamma$ $\supseteq L$ Rule on previous line

This ends the proof of Part 1.

Part 2:

Case: Initial Cut

    Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle\mathcal{E}'}{\Sigma;\Psi \models I' \supseteq I'' \quad (P \text{ atomic})}}{\Sigma;\Psi;\Gamma, A[\![I]\!]; P[I'] \Longrightarrow P[I'']} \; \text{init}$$

$\Sigma;\Psi;\Gamma; P[I'] \Longrightarrow P[I'']$ init Rule on $\mathcal{E}'$

Case: Copy Cut

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta', A[I] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow \gamma} \; \text{copy}$$

$\Sigma; \Psi; \Gamma; \Delta', A[I] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow \gamma$ I.H.(1) on A, $\mathcal{D}$, and previous line

Case: Right Commutative Cuts

    Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma', B[\![I']\!], A[\![I]\!]; \Delta', B[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma', B[\![I']\!], A[\![I]\!]; \Delta' \Longrightarrow \gamma} \; \text{copy}$$

$\Sigma; \Psi; \Gamma', B[\![I']\!]; \Delta', B[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma', B[\![I']\!]; \Delta' \Longrightarrow \gamma$ copy Rule on previous line

    Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta'_1 \Longrightarrow B_1[I']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta'_2 \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta'_1, \Delta'_2 \Longrightarrow B_1 \otimes B_2[I']} \; \otimes R$$

$\Sigma; \Psi; \Gamma; \Delta'_1 \Longrightarrow B_1[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta'_2 \Longrightarrow B_2[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_2$

$\Sigma; \Psi; \Gamma; \Delta'_1, \Delta'_2 \Longrightarrow B_1 \otimes B_2[I']$ $\otimes R$ Rule on previous lines

    Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta'_1, B_1[I'], B_2[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta'_1, B_1 \otimes B_2[I'] \Longrightarrow \gamma} \; \otimes L$$

$\Sigma; \Psi; \Gamma; \Delta'_1, B_1[I'], B_2[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta'_1, B_1 \otimes B_2[I'] \Longrightarrow \gamma$ $\otimes L$ Rule on previous line

    Subcase:

$$\mathcal{E} = \frac{}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \cdot \Longrightarrow \mathbf{1}[I']} \; \mathbf{1}R$$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow \mathbf{1}[I']$ $\mathbf{1}R$ Rule

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Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1' \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', \mathbf{1}[I'] \Longrightarrow \gamma} \ \mathbf{1}L$$

$\Sigma; \Psi; \Gamma; \Delta_1' \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1', \mathbf{1}[I'] \Longrightarrow \gamma$ $\mathbf{1}L$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_1[I']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_1 \mathbin{\&} B_2[I']} \ \&R$$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_1[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_2[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_2$
$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_1 \mathbin{\&} B_2[I']$ $\&R$ Rule on previous lines

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_1[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_1 \mathbin{\&} B_2[I'] \Longrightarrow \gamma} \ \&L_1$$

$\Sigma; \Psi; \Gamma; \Delta_1', B_1[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1', B_1 \mathbin{\&} B_2[I'] \Longrightarrow \gamma$ $\&L_1$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_2[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_1 \mathbin{\&} B_2[I'] \Longrightarrow \gamma} \ \&L_2$$

$\Sigma; \Psi; \Gamma; \Delta_1', B_2[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1', B_1 \mathbin{\&} B_2[I'] \Longrightarrow \gamma$ $\&L_2$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow \top[I']} \ \top R$$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow \top[I']$ $\top R$ Rule

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_1[I']}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_1 \oplus B_2[I']} \ \oplus R_1$$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_1[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_1 \oplus B_2[I']$ $\oplus R_1$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_2[I']}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_1 \oplus B_2[I']} \ \oplus R_2$$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_2[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_1 \oplus B_2[I']$ $\oplus R_2$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_1[I'] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_2[I'] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', B_1 \oplus B_2[I'] \Longrightarrow \gamma} \ \oplus L$$

$\Sigma; \Psi; \Gamma; \Delta_1', B_1[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta_1', B_2[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_2$

$\Sigma; \Psi; \Gamma; \Delta_1', B_1 \oplus B_2[I'] \Longrightarrow \gamma$ $\oplus L$ Rule on previous lines

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma, A[\![I]\!]; \Delta', B_1[i'] \Longrightarrow B_2[i']}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B_1 \multimap B_2[I']} \ \multimap R$$

$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; \cdot \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma, i'\text{:interval}; \Psi, I' \supseteq i'; \Gamma; \Delta', B_1[i'] \Longrightarrow B_2[i']$ I.H.(2) on previous line and $\mathcal{E}'$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B_1 \multimap B_2[I']$ $\multimap R$ Rule on previous line

Subcase: The last rule of $\mathcal{D}$ is $\multimap L$, and $\mathcal{D}$ has the form:

$$\frac{\overset{\mathcal{E}_1}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1' \Longrightarrow B_1[I'']} \quad \overset{\mathcal{E}_2}{\Sigma; \Psi \Longrightarrow I' \supseteq I''} \quad \overset{\mathcal{E}_3}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_2', B_2[I''] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', \Delta_2', B_1 \multimap B_2[I'] \Longrightarrow \gamma} \ \multimap L$$

$\Sigma; \Psi; \Gamma; \Delta_1' \Longrightarrow B_1[I'']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$

$\Sigma; \Psi; \Gamma; \Delta_2', B_2[I''] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_3$

$\Sigma; \Psi; \Gamma; \Delta_1', \Delta_2', B_1 \multimap B_2[I'] \Longrightarrow \gamma$ $\multimap L$ Rule on first line, $\mathcal{E}_2$, and second line

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle \mathcal{E}'}{\Sigma;\Psi;\Gamma,A[\![I]\!];\cdot \Longrightarrow B[I']}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\cdot \Longrightarrow !B[I']} \; !R$$

$\Sigma;\Psi;\Gamma;\cdot \Longrightarrow B[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$

$\Sigma;\Psi;\Gamma;\cdot \Longrightarrow !B[I']$ $!R$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle \mathcal{E}'}{\Sigma;\Psi;\Gamma,A[I],B[\![I']\!];\Delta_1' \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma,A[I];\Delta_1',!B[I'] \Longrightarrow \gamma} \; !L$$

$\Sigma;\Psi;\Gamma,B[\![I']\!];\cdot \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma;\Psi;\Gamma,B[\![I']\!];\Delta_1' \Longrightarrow \gamma$ I.H.(2) on previous line and $\mathcal{E}'$

$\Sigma;\Psi;\Gamma;\Delta_1',!B[I'] \Longrightarrow \gamma$ $!L$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle \mathcal{E}'}{\Sigma,i':\text{interval};\Psi,I' \supseteq i';\Gamma,A[\![I]\!],B_1[\![i']\!];\Delta' \Longrightarrow B_2[i']}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow B_1 \supset B_2[I']} \; \supset R$$

$\Sigma,i':\text{interval};\Psi,I' \supseteq i';\Gamma,B_1[\![i']\!];\cdot \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma,i':\text{interval};\Psi,I' \supseteq i';\Gamma,B_1[\![i']\!];\Delta' \Longrightarrow B_2[i']$ I.H.(2) on previous line and $\mathcal{E}'$

$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow B_1 \supset B_2[I']$ $\supset R$ Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle \mathcal{E}_1}{\Sigma;\Psi;\Gamma,A[\![I]\!];\cdot \Longrightarrow B_1[I'']} \quad \overset{\displaystyle \mathcal{E}_2}{\Sigma;\Psi \Longrightarrow I'' \supseteq I'} \quad \overset{\displaystyle \mathcal{E}_3}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',B_2[I''] \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',B_1 \supset B_2[I'] \Longrightarrow \gamma} \; \supset L$$

$\Sigma;\Psi;\Gamma;\cdot \Longrightarrow B_1[I'']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$

$\Sigma;\Psi;\Gamma;\Delta_1',B_2[I''] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_3$

$\Sigma;\Psi;\Gamma;\Delta_1',B_1 \supset B_2[I'] \Longrightarrow \gamma$ $\supset L$ Rule on first line, $\mathcal{E}_2$, and second line

Subcase:

$$\mathcal{E} = \frac{\overset{\displaystyle \mathcal{E}'}{\Sigma,x{:}s;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow B[I']}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow \forall x{:}s.B[I']} \; \forall R$$

$\Sigma,x{:}s;\Psi;\Gamma;\cdot \Longrightarrow A[I]$ Weakening on $\mathcal{D}$

$\Sigma,x{:}s;\Psi;\Gamma;\Delta' \Longrightarrow B[I']$ I.H.(2) on $\mathcal{D}$, and $\mathcal{E}'$

$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow \forall x{:}s.B[I']$ $\forall R$ Rule on previous line

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',[t/x]B[I'] \Longrightarrow \gamma} \quad \overset{\mathcal{E}_2}{\Sigma \models t{:}s}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',\forall x{:}s.B[I'] \Longrightarrow \gamma} \ \forall L$$

$\Sigma;\Psi;\Gamma;\Delta_1',[t/x]B[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma;\Psi;\Gamma;\Delta_1',\forall x{:}s.B[I'] \Longrightarrow \gamma$ $\forall L$ Rule on previous line and $\mathcal{E}_2$

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow B[I']}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow B \ @ \ I'[I'']} \ @R$$

$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow B[I']$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow B \ @ \ I'[I'']$ $@R$ Rule on previous line

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',B[I'] \Longrightarrow \gamma}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',B \ @ \ I'[I''] \Longrightarrow \gamma} \ @L$$

$\Sigma;\Psi;\Gamma;\Delta_1',B[I'] \Longrightarrow \gamma$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma;\Psi;\Gamma;\Delta_1',B \ @ \ I'[I''] \Longrightarrow \gamma$ $@L$ Rule on previous line

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}'}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow (K \text{ affirms } B) \text{ at } I'}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta' \Longrightarrow \langle K \rangle B[I']} \ \langle\rangle R$$

$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma;\Psi;\Gamma;\Delta' \Longrightarrow \langle K \rangle B[I']$ $\langle\rangle R$ Rule on previous line

Subcase:

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_1}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',B[I'] \Longrightarrow (K \text{ affirms } D) \text{ at } I''} \quad \overset{\mathcal{E}_2}{\Sigma;\Psi \models I' \supseteq I''}}{\Sigma;\Psi;\Gamma,A[\![I]\!];\Delta_1',\langle K \rangle B[I'] \Longrightarrow (K \text{ affirms } D) \text{ at } I''} \ \langle\rangle L$$

$\Sigma;\Psi;\Gamma;\Delta_1',B[I'] \Longrightarrow (K \text{ affirms } D) \text{ at } I''$ I.H.(2) on $\mathcal{D}$ and $\mathcal{E}_1$
$\Sigma;\Psi;\Gamma;\Delta_1',\langle K \rangle B[I'] \Longrightarrow (K \text{ affirms } D) \text{ at } I''$ $\langle\rangle L$ Rule on previous line and $\mathcal{E}_2$

Subcase:

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$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow B[I']}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta' \Longrightarrow (K \text{ affirms } B) \text{ at } I'} \text{ affirms}$$

$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow B[I']$	I.H.(2) on $\mathcal{D}$ and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta' \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	affirms Rule on previous line

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi \models I' \supseteq I''}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \cdot \Longrightarrow I' \supseteq I''[I''']} \supseteq R$$

$\Sigma; \Psi; \Gamma; \cdot \Longrightarrow I' \supseteq I''[I''']$	$\supseteq R$ Rule on $\mathcal{E}'$

Subcase:

$$\mathcal{E} = \frac{\overset{\mathcal{E}'}{\Sigma; \Psi, I' \supseteq I''; \Gamma, A[\![I]\!] \Longrightarrow \gamma}}{\Sigma; \Psi; \Gamma, A[\![I]\!]; \Delta_1', I' \supseteq I''[I'''] \Longrightarrow \gamma} \supseteq L$$

$\Sigma; \Psi, I' \supseteq I''; \Gamma; \cdot \Longrightarrow A[I]$	Weakening on $\mathcal{D}$
$\Sigma; \Psi, I' \supseteq I''; \Gamma; \Delta_1' \Longrightarrow \gamma$	I.H.(2) on previous line and $\mathcal{E}'$
$\Sigma; \Psi; \Gamma; \Delta_1', I' \supseteq I''[I'''] \Longrightarrow \gamma$	$\supseteq L$ Rule on previous line

This ends the proof of Part 2.
Part 3:

Case: "Initial" Cut

    Subcase:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow A[I]}}{\Sigma; \Psi; \Gamma; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \text{ affirms}$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(1) on A, $\mathcal{D}'$, and $\mathcal{E}$

Case: Left Commutative Cuts

    Subcase:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Gamma', D[\![I'']\!]; \Delta, D[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma', D[\![I'']\!]; \Delta \Longrightarrow (K \text{ affirms } A) \text{ at } I} \text{ copy}$$

$\Sigma; \Psi; \Gamma', D[\![I'']\!]; \Delta', \Delta, D[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma', D[\![I'']\!]; \Delta', \Delta \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	copy Rule on previous line

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1, D_1[I''], D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, D_1 \otimes D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \otimes L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1[I''], D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1 \otimes D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\otimes L$ Rule on previous line

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, \mathbf{1}[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \mathbf{1}L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1 \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, \mathbf{1}[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\mathbf{1}L$ Rule on previous line

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1, D_1[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, D_1 \mathbin{\&} D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \&L_1$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1 \mathbin{\&} D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\&L_1$ Rule on previous line

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Gamma; \Delta_1, D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, D_1 \mathbin{\&} D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \&L_2$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1 \mathbin{\&} D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\&L_2$ Rule on previous line

Subcase: The last rule of $\mathcal{D}$ is $\oplus L$, and $\mathcal{D}$ has the form:

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \Sigma; \Psi; \Gamma; \Delta_1, D_1[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I \quad & \Sigma; \Psi; \Gamma; \Delta_1, D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I\end{array}}{\Sigma; \Psi; \Gamma; \Delta_1, D_1 \oplus D_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \ \oplus L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}_1$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}_2$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D_1 \oplus D_2[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\oplus L$ Rule on previous lines

Subcase: The last rule of $\mathcal{D}$ is $\multimap L$, and $\mathcal{D}$ has the form:

$$\dfrac{\overset{\mathcal{D}_1}{\Sigma;\Psi;\Gamma;\Delta_1 \Longrightarrow D_1[I_3]} \quad \overset{\mathcal{D}_2}{\Sigma;\Psi \models I_2 \supseteq I_3} \quad \overset{\mathcal{D}_3}{\Sigma;\Psi;\Gamma;\Delta_2, D_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma;\Psi;\Gamma;\Delta_1,\Delta_2, D_1 \multimap D_2[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \multimap L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_2, D_2[I_3] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}_3$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma;\Psi;\Gamma;\Delta_1,\Delta',\Delta_2, D_1 \multimap D_2[I_2] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\multimap L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma;\Psi;\Gamma, D[\![I'']\!];\Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma;\Psi;\Gamma;\Delta_1, !D[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \, !L$$

$\Sigma;\Psi;\Gamma, D[\![I'']\!];\Delta', A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	Weakening on $\mathcal{E}$
$\Sigma;\Psi;\Gamma, D[\![I'']\!];\Delta',\Delta_1 \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, previous line, and $\mathcal{F}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, !D[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$!L$ Rule on previous line

Subcase: The last of rule of $\mathcal{D}$ is $\supset L$, and $\mathcal{D}$ has the form:

$$\dfrac{\overset{\mathcal{D}_1}{\Sigma;\Psi;\Gamma;\cdot \Longrightarrow D_1[I_3]} \quad \overset{\mathcal{D}_2}{\Sigma;\Psi \models I_2 \supseteq I_3} \quad \overset{\mathcal{D}_3}{\Sigma;\Psi;\Gamma;\Delta_1, D_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma;\Psi;\Gamma;\Delta_1, D_1 \supset D_2[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \supset L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, D_2[I_3] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}_3$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, D_1 \supset D_2[I_2] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\supset L$ Rule on $\mathcal{D}_1$, $\mathcal{D}_2$, and previous line

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma;\Psi;\Gamma;\Delta_1, [t/x]B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \quad \overset{\mathcal{D}_2}{\Sigma \models t{:}s}}{\Sigma;\Psi;\Gamma;\Delta_1, \forall x{:}s.B[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \, \forall L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, [t/x]B[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}_1$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, \forall x{:}s.B[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$\forall L$ Rule on previous line and $\mathcal{D}_2$

Subcase:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Sigma;\Psi;\Gamma;\Delta_1, D[I_2] \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma;\Psi;\Gamma;\Delta_1, D @ I_2[I_3] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \, @L$$

$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, D[I_2] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$
$\Sigma;\Psi;\Gamma;\Delta',\Delta_1, D @ I_2[I_3] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$	$@L$ Rule on previous line

Subcase:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Gamma; \Delta_1, D[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I'' \supseteq I}}{\Sigma; \Psi; \Gamma; \Delta_1, \langle K \rangle D[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \langle \rangle L$$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, D[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ I.H.(3) on A, $\mathcal{D}'$, $\mathcal{E}$, and $\mathcal{F}$

$\Sigma; \Psi \models I'' \supseteq I'$ Transitivity Property of $\models$ on $\mathcal{D}_2$ and $\mathcal{F}$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, \langle K \rangle D[I''] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ $\langle \rangle L$ Rule on previous lines

Subcase:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi, I_2 \supseteq I_3; \Gamma; \Delta_1 \Longrightarrow (K \text{ affirms } A) \text{ at } I}}{\Sigma; \Psi; \Gamma; \Delta_1, I_2 \supseteq I_3[I_4] \Longrightarrow (K \text{ affirms } A) \text{ at } I} \; \supseteq L$$

$\Sigma; \Psi, I_2 \supseteq I_3; \Gamma; \Delta', A[I] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ Weakening on $\mathcal{E}$

$\Sigma; \Psi, I_2 \supseteq I_3; \Gamma; \Delta', \Delta_1 \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ I.H.(3) on A, $\mathcal{D}'$, previous line, and $\mathcal{F}$

$\Sigma; \Psi; \Gamma; \Delta', \Delta_1, I_2 \supseteq I_3[I_4] \Longrightarrow (K \text{ affirms } B) \text{ at } I'$ $\supseteq L$ Rule on previous line

This ends the proof of part 3.

$\square$

C   Enforcement of a Fragment of η-logic in ζ-logic

C.1   Translation From ζ-logic to η-logic

Theorem 4. Suppose $\Sigma; \Psi \models I''' \supseteq I''$ for each $I''' \in \vec{I}$ and for each $I''' \in \vec{I'}$. Then,

1. If $\Xi; \Theta; \Lambda \Longrightarrow F$ in ζ-logic, then $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow F[I'']$ in η-logic.
2. If $\Xi; \Theta; \Lambda \Longrightarrow K$ affirms F in ζ-logic, then $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } F) \text{ at } I''$ in η-logic.

Proof. By simultaneous structural induction on the first given derivation, $\Xi; \Theta; \Lambda \Longrightarrow A$ or $\Xi; \Theta; \Lambda \Longrightarrow K$ affirms A.

    Part 1:

Case:

$$\mathcal{D} = \frac{}{\Xi; \Theta; A \Longrightarrow A} \; \text{init}$$

Let $\vec{I'} = \{I'\}$.

$\Sigma; \Psi \models I' \supseteq I''$ Containment assumption for $\vec{I'}$ and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; A[I'] \Longrightarrow A[I'']$ init Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Xi; \Theta_1, B; \Lambda, B \Longrightarrow A}}{\Xi; \Theta_1, B; \Lambda \Longrightarrow A} \text{ copy}$$

Let $\vec{I} = \vec{I_1} \cup \{I\}$.

$\Sigma; \Psi \models I \supseteq I''$	Containment assumption for $\vec{I}$ and previous line
$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'} \cup \{I\}$	Containment assumption for $\vec{I'}$ and previous line
$\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[\![I]\!]; \Lambda[\vec{I'}], B[I] \Longrightarrow A[I'']$	I.H.(1) on $\mathcal{D}'$, containment assumption for $\vec{I}$, and previous line
$\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[\![I]\!]; \Lambda[\vec{I'}] \Longrightarrow A[I'']$	copy Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Xi; \Theta; \Lambda_1 \Longrightarrow A_1} \quad \overset{\mathcal{D}_2}{\Xi; \Theta; \Lambda_2 \Longrightarrow A_2}}{\Xi; \Theta; \Lambda_1, \Lambda_2 \Longrightarrow A_1 \otimes A_2} \otimes R$$

Let $\vec{I'} = \vec{I'_1} \cup \vec{I'_2}$.

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_1}$	Containment assumption for $\vec{I'}$ and previous line
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}] \Longrightarrow A_1[I'']$	I.H.(1) on $\mathcal{D}_1$, containment assumption for $\vec{I}$, and previous line
$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_2}$	Containment assumption for $\vec{I'}$ and first line
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_2[\vec{I'_2}] \Longrightarrow A_2[I'']$	I.H.(1) on $\mathcal{D}_2$, containment assumption for $\vec{I}$, and previous line
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], \Lambda_2[\vec{I'_2}] \Longrightarrow A_1 \otimes A_2[I'']$	$\otimes R$ Rule on third and fifth lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda_1, B_1, B_2 \Longrightarrow A}}{\Xi; \Theta; \Lambda_1, B_1 \otimes B_2 \Longrightarrow A} \otimes L$$

Let $\vec{I'} = \vec{I'_1} \cup \{I'\}$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1[I'], B_2[I'] \Longrightarrow A[I'']$	I.H.(1) on $\mathcal{D}'$ and containment assumptions for $\vec{I}$ and $\vec{I'}$
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \otimes B_2[I'] \Longrightarrow A[I'']$	$\otimes L$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{}{\overline{\Xi; \Theta; \cdot \Longrightarrow \mathbf{1}}} \ \mathbf{1}R$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow \mathbf{1}[I'']$ $\mathbf{1}R$ Rule

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c} \mathcal{D}' \\ \Xi; \Theta; \Lambda_1 \Longrightarrow A \end{array}}{\Xi; \Theta; \Lambda_1, \mathbf{1} \Longrightarrow A} \ \mathbf{1}L$$

Let $\vec{I}' = \vec{I'_1} \cup \{I'\}$.

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_1}$	Containment assumption for $\vec{I}'$ and previous line
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}] \Longrightarrow A[I'']$	I.H.(1) on $\mathcal{D}'$, containment assumption for $\vec{I}$, and previous line
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], \mathbf{1}[I'] \Longrightarrow A[I'']$	$\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ \Xi; \Theta; \Lambda \Longrightarrow A_1 & \Xi; \Theta; \Lambda \Longrightarrow A_2 \end{array}}{\Xi; \Theta; \Lambda \Longrightarrow A_1 \ \& \ A_2} \ \&R$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I}'] \Longrightarrow A_1[I'']$	I.H.(1) on $\mathcal{D}_1$ and containment assumptions for $\vec{I}$ and $\vec{I}'$
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I}'] \Longrightarrow A_2[I'']$	I.H.(1) on $\mathcal{D}_2$ and containment assumptions for $\vec{I}$ and $\vec{I}'$
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I}'] \Longrightarrow A_1 \ \& \ A_2[I'']$	$\&R$ Rule on previous lines

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c} \mathcal{D}' \\ \Xi; \Theta; \Lambda_1, B_1 \Longrightarrow A \end{array}}{\Xi; \Theta; \Lambda_1, B_1 \ \& \ B_2 \Longrightarrow A} \ \&L_1$$

Let $\vec{I}' = \vec{I'_1}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1[I'] \Longrightarrow A[I'']$	I.H.(1) on $\mathcal{D}'$ and containment assumptions for $\vec{I}$ and $\vec{I}'$
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \ \& \ B_2[I'] \Longrightarrow A[I'']$	$\&L_1$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow A}}{\Xi; \Theta; \Lambda_1, B_1 \& B_2 \Longrightarrow A} \; \& L_2$$

Let $\vec{I'} = \vec{I'_1}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_2[I'] \Longrightarrow A[I'']$ I.H.(1) on $\mathcal{D}'$ and containment assumptions

                                                    for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \& B_2[I'] \Longrightarrow A[I'']$ $\& L_2$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{}{\Xi; \Theta; \Lambda \Longrightarrow \top} \; \top R$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow \top[I'']$ $\top R$ Rule

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda \Longrightarrow A_1}}{\Xi; \Theta; \Lambda \Longrightarrow A_1 \oplus A_2} \; \oplus R_1$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1[I'']$ I.H.(1) on $\mathcal{D}'$ and containment assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \oplus A_2[I'']$ $\oplus R_1$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda \Longrightarrow A_2}}{\Xi; \Theta; \Lambda \Longrightarrow A_1 \oplus A_2} \; \oplus R_2$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_2[I'']$ I.H.(1) on $\mathcal{D}'$ and containment assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \oplus A_2[I'']$ $\oplus R_2$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow A} \quad \overset{\mathcal{D}_2}{\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow A}}{\Xi; \Theta; \Lambda_1, B_1 \oplus B_2 \Longrightarrow A} \; \oplus L$$

Let $\vec{I'} = \vec{I'_1}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1[I'] \Longrightarrow A[I'']$ 　　　　I.H.(1) on $\mathcal{D}_1$ and containment assumptions
　　　　　　　　　　　　　　　　　　　　　　　　　　　for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_2[I'] \Longrightarrow A[I'']$ 　　　　I.H.(1) on $\mathcal{D}_2$ and containment assumptions
　　　　　　　　　　　　　　　　　　　　　　　　　　　for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \oplus B_2[I'] \Longrightarrow A[I'']$ 　　　　$\oplus L$ Rule on previous lines

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Xi; \Theta; \Lambda, A_1 \Longrightarrow A_2\end{array}}{\Xi; \Theta; \Lambda \Longrightarrow A_1 \multimap A_2} \; \multimap R$$

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I}$ 　　　　Containment assumption for $\vec{I}$

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq I''$ for all $I''' \in \vec{I}$ 　　　　Weakening Property of $\models$ on previous
　　　　　　　　　　　　　　　　　　　　　　　　　　　line

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models I'' \supseteq i''$ 　　　　Hypothesis Property of $\models$

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq i''$ for all $I''' \in \vec{I}$ 　　　　Transitivity Property of $\models$ on second
　　　　　　　　　　　　　　　　　　　　　　　　　　　and third lines

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'}$ 　　　　Containment assumption for $\vec{I'}$

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq I''$ for all $I''' \in \vec{I'}$ 　　　　Weakening Property of $\models$ on previous
　　　　　　　　　　　　　　　　　　　　　　　　　　　line

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq i''$ for all $I''' \in \vec{I'}$ 　　　　Transitivity Property of $\models$ on sixth
　　　　　　　　　　　　　　　　　　　　　　　　　　　and third lines

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models i'' \supseteq i''$ 　　　　Reflexivity Property of $\models$

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq i''$ for all $I''' \in \vec{I'} \cup \{i''\}$ 　　　　Seventh and eighth lines

$\Sigma, i''{:}\text{interval}; \Psi, I'' \supseteq i''; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}], A_1[i''] \Longrightarrow A_2[i'']$ 　　　　I.H.(1) on $\mathcal{D}'$, fourth line,
　　　　　　　　　　　　　　　　　　　　　　　　　　　and ninth line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \multimap A_2[I'']$ 　　　　$\multimap R$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Xi; \Theta; \Lambda_1 \Longrightarrow B_1 & \Xi; \Theta; \Lambda_2, B_2 \Longrightarrow A\end{array}}{\Xi; \Theta; \Lambda_1, \Lambda_2, B_1 \multimap B_2 \Longrightarrow A} \; \multimap L$$

Let $\vec{I'} = \vec{I'_1} \cup \vec{I'_2} \cup \{I'\}$.

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_1}$ 　　　　Containment assumption for $\vec{I'}$
　　　　　　　　　　　　　　　　　　　　　　　　　　　and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}] \Longrightarrow B_1[I'']$ 　　　　I.H.(1) on $\mathcal{D}_1$, containment assumption for $\vec{I}$,
　　　　　　　　　　　　　　　　　　　　　　　　　　　and previous line

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_2}$ 　　　　Containment assumption for $\vec{I'}$ and first line

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$\Sigma; \Psi \models I'' \supseteq I''$ Reflexivity Property of $\models$

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I}_2^I \cup \{I''\}$ Fourth and fifth lines

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_2[\vec{I}_2^I], B_2[I''] \Longrightarrow A[I'']$ I.H.(1) on $\mathcal{D}_2$, containment assumption for $\vec{I}$, and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I}_1^I], \Lambda_2[\vec{I}_2^I], B_1 \multimap B_2[I'] \Longrightarrow A[I'']$ $\multimap L$ Rule on third, fifth, and seventh lines

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\ \Xi; \Theta; \cdot \Longrightarrow A_1\end{array}}{\Xi; \Theta; \cdot \Longrightarrow !A_1} \ !R$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow A_1[I'']$ I.H.(1) on $\mathcal{D}'$ and containment assumption for $\vec{I}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow !A_1[I'']$ $!R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\ \Xi; \Theta, B; \Lambda_1 \Longrightarrow A\end{array}}{\Xi; \Theta; \Lambda_1, !B \Longrightarrow A} \ !L$$

Let $\vec{I'} = \vec{I}_1^I \cup \{I'\}$.

$\Sigma; \Psi \models I' \supseteq I''$ Containment assumption for $\vec{I'}$ and previous line

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I} \cup \{I'\}$ Containment assumption for $\vec{I}$ and previous line

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I}_1^I$ Containment assumption for $\vec{I'}$ and first line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!], B[I']; \Lambda_1[\vec{I}_1^I] \Longrightarrow A[I'']$ I.H.(1) on $\mathcal{D}'$, third line, and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I}_1^I], !B[I'] \Longrightarrow A[I'']$ $!L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}'\\ \Xi; \Theta, A_1; \Lambda \Longrightarrow A_2\end{array}}{\Xi; \Theta; \Lambda \Longrightarrow A_1 \supset A_2} \ \supset R$$

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I}$ Containment assumption for $\vec{I}$

$\Sigma, i''{:}\mathrm{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq I''$ for all $I''' \in \vec{I}$ Weakening Property of $\models$ on previous line

$\Sigma, i''{:}\mathrm{interval}; \Psi, I'' \supseteq i'' \models I'' \supseteq i''$ Hypothesis Property of $\models$

$\Sigma, i''{:}\mathrm{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq i''$ for all $I''' \in \vec{I}$ Transitivity Property of $\models$ on second and third lines

$\Sigma, i''{:}\mathrm{interval}; \Psi, I'' \supseteq i'' \models i'' \supseteq i''$ Reflexivity Property of $\models$

$\Sigma, i''{:}\mathrm{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq i''$ for all $I''' \in \vec{I} \cup \{i''\}$ Fourth and fifth lines

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'}$ Containment assumption for $\vec{I'}$

$\Sigma, i''{:}\mathsf{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq I''$ for all $I''' \in \vec{I'}$ Weakening Property of $\models$ on previous line

$\Sigma, i''{:}\mathsf{interval}; \Psi, I'' \supseteq i'' \models I''' \supseteq i''$ for all $I''' \in \vec{I'}$ Transitivity Property of $\models$ on eighth and third lines

$\Sigma, i''{:}\mathsf{interval}; \Psi, I'' \supseteq i''; \Theta[\![\vec{I}]\!], A_1[\![i'']\!]; \Lambda[\vec{I'}] \Longrightarrow A_2[i'']$ I.H.(1) on $\mathcal{D'}$, sixth line, and third line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \supset A_2[I'']$ $\supset R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Xi; \Theta; \cdot \Longrightarrow B_1} \quad \overset{\mathcal{D}_2}{\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow A}}{\Xi; \Theta; \Lambda_1, B_1 \supset B_2 \Longrightarrow A} \supset L$$

Let $\vec{I'} = \vec{I'_1} \cup \{I'\}$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow B_1[I'']$ I.H.(1) on $\mathcal{D}_1$ and containment assumption for $\vec{I}$

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_1}$ Containment assumption for $\vec{I'}$ and first line

$\Sigma; \Psi \models I'' \supseteq I''$ Reflexivity Property of $\models$

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'_1} \cup \{I''\}$ Third and fourth lines

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_2[I''] \Longrightarrow A[I'']$ I.H.(1) on $\mathcal{D}_2$, containment assumption for $\vec{I}$, and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \supset B_2[I'] \Longrightarrow A[I'']$ $\supset L$ Rule on second, fourth, and sixth lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D'}}{\Xi, x{:}s; \Theta; \Lambda \Longrightarrow A_1}}{\Xi; \Theta; \Lambda \Longrightarrow \forall x{:}s.A_1} \forall R$$

$\Sigma, x{:}s; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I}$ Weakening Property of $\models$ on containment assumption for $\vec{I}$

$\Sigma, x{:}s; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'}$ Weakening Property of $\models$ on containment assumption for $\vec{I'}$

$\Sigma, x{:}s; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1[I'']$ I.H.(1) on $\mathcal{D'}$ and previous lines

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow \forall x{:}s.A_1[I'']$ $\forall R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Xi; \Theta; \Lambda_1, [t/x]B \Longrightarrow A} \quad \overset{\mathcal{D}_2}{\Xi \vdash t{:}s}}{\Xi; \Theta; \Lambda_1, \forall x{:}s.B \Longrightarrow A} \forall L$$

Let $\vec{I'} = \vec{I'_1}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], [t/x]B[I'] \Longrightarrow A[I'']$ I.H.(1) on $\mathcal{D}_1$ and containment assumption

         for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], \forall x{:}s.B[I'] \Longrightarrow A[I'']$ $\forall L$ Rule on previous line and $\mathcal{D}_2$

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}'\\\Xi; \Theta; \Lambda \Longrightarrow K \text{ affirms } A_1\end{array}}{\Xi; \Theta; \Lambda \Longrightarrow \langle K \rangle A_1} \; \langle\rangle R$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } A_1) \text{ at } I''$ I.H.(2) on $\mathcal{D}'$ and containment assumptions

         for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow \langle K \rangle A_1[I'']$ $\langle\rangle R$ Rule on previous line

Part 2:

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}'\\\Xi; \Theta_1, B; \Lambda, B \Longrightarrow K \text{ affirms } A\end{array}}{\Xi; \Theta_1, B; \Lambda \Longrightarrow K \text{ affirms } A} \; \text{copy}$$

Let $\vec{I} = \vec{I_1} \cup \{I\}$.

$\Sigma; \Psi \models I \supseteq I''$ Containment assumption for $\vec{I}$ and previous line

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I'} \cup \{I\}$ Containment assumption for $\vec{I'}$ and previous line

$\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[I]; \Lambda[\vec{I'}], B[I] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}'$, containment

         assumption for $\vec{I}$, and previous line

$\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[I]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ copy Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}'\\\Xi; \Theta; \Lambda_1, B_1, B_2 \Longrightarrow K \text{ affirms } A\end{array}}{\Xi; \Theta; \Lambda_1, B_1 \otimes B_2 \Longrightarrow K \text{ affirms } A} \; \otimes L$$

Let $\vec{I'} = \vec{I'_1} \cup \{I'\}$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1[I'], B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}'$ and containment

         assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \otimes B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\otimes L$ Rule on previous line

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Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda_1 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, \mathbf{1} \Longrightarrow K \text{ affirms } A} \ \mathbf{1}L$$

Let $\vec{I'} = \vec{I_1'} \cup \{I'\}$.

$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I_1'}$ Containment assumption for $\vec{I'}$ and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}'$, containment assumption for $\vec{I}$,

and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \mathbf{1}[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, B_1 \ \& \ B_2 \Longrightarrow K \text{ affirms } A} \ \& L_1$$

Let $\vec{I'} = \vec{I_1'}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}'$ and containment

assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \ \& \ B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\& L_1$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, B_1 \ \& \ B_2 \Longrightarrow K \text{ affirms } A} \ \& L_2$$

Let $\vec{I'} = \vec{I_1'}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}'$ and containment

assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \ \& \ B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\& L_2$ Rule on previous line

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow K \text{ affirms } A} \quad \overset{\mathcal{D}_2}{\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, B_1 \oplus B_2 \Longrightarrow K \text{ affirms } A} \ \oplus L$$

Let $\vec{I'} = \vec{I_1'}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}_1$ and containment assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}_2$ and containment assumptions for $\vec{I}$ and $\vec{I'}$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \oplus B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\oplus L$ Rule on previous lines

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Xi; \Theta; \Lambda_1 \Longrightarrow B_1} \quad \overset{\mathcal{D}_2}{\Xi; \Theta; \Lambda_2, B_2 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, \Lambda_2, B_1 \multimap B_2 \Longrightarrow K \text{ affirms } A} \; \multimap L$$

Let $\vec{I'} = \vec{I_1'} \cup \vec{I_2'} \cup \{I'\}$.

$\Sigma; \Psi \models I''' \supseteq I'' \text{ for all } I''' \in \vec{I_1'}$ Containment assumption for $\vec{I'}$ and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}] \Longrightarrow B_1[I'']$ I.H.(1) on $\mathcal{D}_1$, containment assumption for $\vec{I}$, and previous line

$\Sigma; \Psi \models I''' \supseteq I'' \text{ for all } I''' \in \vec{I_2'}$ Containment assumption for $\vec{I'}$ and first line

$\Sigma; \Psi \models I'' \supseteq I''$ Reflexivity Property of $\models$

$\Sigma; \Psi \models I''' \supseteq I'' \text{ for all } I''' \in \vec{I_2'} \cup \{I''\}$ Fourth and fifth lines

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_2[\vec{I_2'}], B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D}_2$, containment assumption for $\vec{I}$, and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \Lambda_2[\vec{I_2'}], B_1 \multimap B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\multimap L$ Rule on third, fifth, and seventh lines

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D'}}{\Xi; \Theta, B; \Lambda_1 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, !B \Longrightarrow K \text{ affirms } A} \; !L$$

Let $\vec{I'} = \vec{I_1'} \cup \{I'\}$.

$\Sigma; \Psi \models I' \supseteq I''$ Containment assumption for $\vec{I'}$ and previous line

$\Sigma; \Psi \models I''' \supseteq I'' \text{ for all } I''' \in \vec{I} \cup \{I'\}$ Containment assumption for $\vec{I}$ and previous line

$\Sigma; \Psi \models I''' \supseteq I'' \text{ for all } I''' \in \vec{I_1'}$ Containment assumption for $\vec{I'}$ and first line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!], B[\![I']\!]; \Lambda_1[\vec{I_1'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(1) on $\mathcal{D'}$, third line, and previous line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], !B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $!L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Xi; \Theta; \cdot \Longrightarrow B_1} \quad \overset{\mathcal{D}_2}{\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, B_1 \supset B_2 \Longrightarrow K \text{ affirms } A} \supset L$$

Let $\vec{I'} = \vec{I_1'} \cup \{I'\}$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow B_1[I'']$	I.H.(1) on $\mathcal{D}_1$ and containment assumption for $\vec{I}$
$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I_1'}$	Containment assumption for $\vec{I'}$ and first line
$\Sigma; \Psi \models I'' \supseteq I''$	Reflexivity Property of $\models$
$\Sigma; \Psi \models I''' \supseteq I''$ for all $I''' \in \vec{I_1'} \cup \{I''\}$	Third and fourth lines
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_2[I''] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$	I.H.(1) on $\mathcal{D}_2$, containment assumption for $\vec{I}$, and previous line
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \supset B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$	$\supset L$ Rule on second, fourth, and sixth lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Xi; \Theta; \Lambda_1, [t/x]B \Longrightarrow K \text{ affirms } A} \quad \overset{\mathcal{D}_2}{\Xi \vdash t{:}s}}{\Xi; \Theta; \Lambda_1, \forall x{:}s.B \Longrightarrow K \text{ affirms } A} \forall L$$

Let $\vec{I'} = \vec{I_1'}, I'$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], [t/x]B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$	I.H.(1) on $\mathcal{D}_1$ and containment assumption for $\vec{I}$ and $\vec{I'}$
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \forall x{:}s.B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$	$\forall L$ Rule on previous line and $\mathcal{D}_2$

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda \Longrightarrow A}}{\Xi; \Theta; \Lambda \Longrightarrow K \text{ affirms } A} \text{ affirms}$$

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A[I'']$	I.H.(1) on $\mathcal{D}'$ and containment assumptions for $\vec{I}$ and $\vec{I'}$
$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$	affirms Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Xi; \Theta; \Lambda_1, B \Longrightarrow K \text{ affirms } A}}{\Xi; \Theta; \Lambda_1, \langle K \rangle B \Longrightarrow K \text{ affirms } A} \langle \rangle L$$

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Let $\vec{I}' = \vec{I_1'} \cup \{I'\}$.

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ I.H.(2) on $\mathcal{D}'$ and containment
                                                     assumptions for $\vec{I}$ and $\vec{I}'$

$\Sigma; \Psi \models I' \supseteq I''$ Containment assumption for $\vec{I}'$ and first line

$\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \langle K \rangle B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''$ $\langle \rangle L$ Rule on second and third lines

$\square$

C.2   Translation From a Fragment of η-logic to ζ-logic

Theorem 5.

1. If $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I}'] \Longrightarrow F[I'']$, then $\Xi; \Theta; \Lambda \Longrightarrow F$ in ζ-logic.
2. If $\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I}'] \Longrightarrow (K \text{ affirms } F) \text{ at } I''$, then $\Xi; \Theta; \Lambda \Longrightarrow K \text{ affirms } F$ in ζ-logic.

Proof. By simultaneous structural induction on the first given derivation, $\mathcal{D}$.

   Part 1:

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi \models I' \supseteq I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; A[I'] \Longrightarrow A[I'']} \text{ init}$$

$\Xi; \Theta; A \Longrightarrow A$ init Rule

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[\![I]\!]; \Lambda[\vec{I}'], B[I] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[\![I]\!]; \Lambda[\vec{I}'] \Longrightarrow A[I'']} \text{ copy}$$

$\Xi; \Theta_1, B; \Lambda, B \Longrightarrow A$ I.H.(1) on $\mathcal{D}'$

$\Xi; \Theta_1, B; \Lambda \Longrightarrow A$ copy Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}] \Longrightarrow A_1[I'']} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_2[\vec{I_2'}] \Longrightarrow A_2[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \Lambda_2[\vec{I_2'}] \Longrightarrow A_1 \otimes A_2[I'']} \otimes R$$

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$\Xi; \Theta; \Lambda_1 \Longrightarrow A_1$
$\hspace{8cm}$ I.H.(1) on $\mathcal{D}_1$

$\Xi; \Theta; \Lambda_2 \Longrightarrow A_2$
$\hspace{8cm}$ I.H.(1) on $\mathcal{D}_2$

$\Xi; \Theta; \Lambda_1, \Lambda_2 \Longrightarrow A_1 \otimes A_2$
$\hspace{6cm}$ $\otimes R$ Rule on previous lines

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'], B_2[I'] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \otimes B_2[I'] \Longrightarrow A[I'']} \ \otimes L$$

$\Xi; \Theta; \Lambda_1, B_1, B_2 \Longrightarrow A$
$\hspace{7cm}$ I.H.(1) on $\mathcal{D}'$

$\Xi; \Theta; \Lambda_1, B_1 \otimes B_2 \Longrightarrow A$
$\hspace{5.5cm}$ $\otimes L$ Rule on previous line

Case:

$$\mathcal{D} = \cfrac{}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow \mathbf{1}[I'']} \ \mathbf{1}R$$

$\Xi; \Theta; \cdot \Longrightarrow \mathbf{1}$
$\hspace{9cm}$ $\mathbf{1}R$ Rule

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \mathbf{1}[I'] \Longrightarrow A[I'']} \ \mathbf{1}L$$

$\Xi; \Theta; \Lambda_1 \Longrightarrow A$
$\hspace{8cm}$ I.H.(1) on $\mathcal{D}'$

$\Xi; \Theta; \Lambda_1, \mathbf{1} \Longrightarrow A$
$\hspace{6cm}$ $\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1[I'']} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_2[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \ \& \ A_2[I'']} \ \& R$$

$\Xi; \Theta; \Lambda \Longrightarrow A_1$
$\hspace{8cm}$ I.H.(1) on $\mathcal{D}_1$

$\Xi; \Theta; \Lambda \Longrightarrow A_2$
$\hspace{8cm}$ I.H.(1) on $\mathcal{D}_2$

$\Xi; \Theta; \Lambda \Longrightarrow A_1 \ \& \ A_2$
$\hspace{6cm}$ $\& R$ Rule on previous lines

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \,\&\, B_2[I'] \Longrightarrow A[I'']} \,\&L_1$$

$\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow A$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, B_1 \,\&\, B_2 \Longrightarrow A$ $\&L_1$ Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_2[I'] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \,\&\, B_2[I'] \Longrightarrow A[I'']} \,\&L_2$$

$\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow A$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, B_1 \,\&\, B_2 \Longrightarrow A$ $\&L_2$ Rule on previous line

Case:

$$\mathcal{D} = \cfrac{}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow \top[I'']} \,\top R$$

$\Xi; \Theta; \Lambda \Longrightarrow \mathbf{1}$ $\top R$ Rule

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \oplus A_2[I'']} \,\oplus R_1$$

$\Xi; \Theta; \Lambda \Longrightarrow A_1$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda \Longrightarrow A_1 \oplus A_2$ $\oplus R_1$ Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_2[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \oplus A_2[I'']} \,\oplus R_2$$

$$\Xi; \Theta; \Lambda \Longrightarrow A_2 \qquad\qquad \text{I.H.(1) on } \mathcal{D}'$$
$$\Xi; \Theta; \Lambda \Longrightarrow A_1 \oplus A_2 \qquad\qquad \oplus R_2 \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1[I'] \Longrightarrow A[I'']} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_2[I'] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], B_1 \oplus B_2[I'] \Longrightarrow A[I'']} \; \oplus L$$

$$\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow A \qquad\qquad \text{I.H.(1) on } \mathcal{D}_1$$
$$\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow A \qquad\qquad \text{I.H.(1) on } \mathcal{D}_2$$
$$\Xi; \Theta; \Lambda_1, B_1 \oplus B_2 \Longrightarrow A \qquad\qquad \oplus L \text{ Rule on previous lines}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma, i''\text{:interval}; \Psi, I'' \supseteq i''; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}], A_1[i''] \Longrightarrow A_2[i'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \multimap A_2[I'']} \; \multimap R$$

$$\Xi; \Theta; \Lambda, A_1 \Longrightarrow A_2 \qquad\qquad \text{I.H.(1) on } \mathcal{D}'$$
$$\Xi; \Theta; \Lambda \Longrightarrow A_1 \multimap A_2 \qquad\qquad \multimap R \text{ Rule on previous line}$$

Case: The last rule of $\mathcal{D}$ is $\multimap L$, and $\mathcal{D}$ has the form:

$$\cfrac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}] \Longrightarrow B_1[I''']} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi \models I' \supseteq I'''} \quad \overset{\mathcal{D}_3}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_2[\vec{I'_2}], B_2[I'''] \Longrightarrow A[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I'_1}], \Lambda_2[\vec{I'_2}], B_1 \multimap B_2[I'] \Longrightarrow A[I'']}$$

$$\Xi; \Theta; \Lambda_1 \Longrightarrow B_1 \qquad\qquad \text{I.H.(1) on } \mathcal{D}_1$$
$$\Xi; \Theta; \Lambda_2, B_2 \Longrightarrow A \qquad\qquad \text{I.H.(1) on } \mathcal{D}_3$$
$$\Xi; \Theta; \Lambda_1, \Lambda_2, B_1 \multimap B_2 \Longrightarrow A \qquad\qquad \multimap L \text{ Rule on previous lines}$$

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow A_1[I'']}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow \,!A_1[I'']} \; !R$$

$$\Xi; \Theta; \cdot \Longrightarrow A_1 \qquad\qquad \text{I.H.(1) on } \mathcal{D}'$$
$$\Xi; \Theta; \cdot \Longrightarrow \,!A_1 \qquad\qquad !R \text{ Rule on previous line}$$

Case:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma; \Psi; \Theta[\![\vec{I}]\!], B[\![I']\!]; \Lambda_1[\vec{I_1}] \Longrightarrow A[I''] \end{array}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1}], !B[I'] \Longrightarrow A[I'']} \ !L$$

$\Xi; \Theta, B; \Lambda_1 \Longrightarrow A$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, !B \Longrightarrow A$ $!L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma, i''{:}\mathsf{interval}; \Psi, I'' \supseteq i''; \Theta[\![\vec{I}]\!], A_1[\![i'']\!]; \Lambda[\vec{I'}] \Longrightarrow A_2[i''] \end{array}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1 \supset A_2[I'']} \ \supset R$$

$\Xi; \Theta, A_1; \Lambda \Longrightarrow A_2$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda \Longrightarrow A_1 \supset A_2$ $\supset R$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\begin{array}{ccc} \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ \Sigma; \Psi; \Theta[\![\vec{I}]\!]; \cdot \Longrightarrow B_1[I'''] & \Sigma; \Psi \models I' \supseteq I''' & \Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1}], B_2[I'''] \Longrightarrow A[I''] \end{array}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1}], B_1 \supset B_2[I'] \Longrightarrow A[I'']} \ \supset L$$

$\Xi; \Theta; \cdot \Longrightarrow B_1$ I.H.(1) on $\mathcal{D}_1$
$\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow A$ I.H.(1) on $\mathcal{D}_3$
$\Xi; \Theta; \Lambda_1, B_1 \supset B_2 \Longrightarrow A$ $\supset L$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\begin{array}{c} \mathcal{D}' \\ \Sigma, x{:}s; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A_1[I''] \end{array}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow \forall x{:}s.A_1[I'']} \ \forall R$$

$\Xi, x{:}s; \Theta; \Lambda \Longrightarrow A_1$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda \Longrightarrow \forall x{:}s.A_1$ $\forall R$ Rule on previous line

Case:

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$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], [t/x]B[I'] \Longrightarrow A_1[I''] \quad \overset{\mathcal{D}_2}{\Sigma \vdash t{:}s}}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \forall x{:}s.B[I'] \Longrightarrow A[I'']} \ \forall L$$

$\Xi; \Theta; \Lambda_1, [t/x]B \Longrightarrow A$	I.H.(1) on $\mathcal{D}_1$
$\Xi \vdash t{:}s$	$\mathcal{D}_2$ with s that is not interval
$\Xi; \Theta; \Lambda_1, \forall x{:}s.B \Longrightarrow A$	$\forall L$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } A_1) \text{ at } I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow \langle K \rangle A_1[I'']} \ \langle\rangle R$$

$\Xi; \Theta; \Lambda \Longrightarrow K \text{ affirms } A_1$	I.H.(2) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda \Longrightarrow \langle K \rangle A_1$	$\langle\rangle R$ Rule on previous line

Part 2:

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[\![I]\!]; \Lambda[\vec{I'}], B[I] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma; \Psi; \Theta_1[\![\vec{I_1}]\!], B[\![I]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \ \text{copy}$$

$\Xi; \Theta_1, B; \Lambda, B \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}'$
$\Xi; \Theta_1, B; \Lambda \Longrightarrow K \text{ affirms } A$	copy Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'], B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \otimes B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \ \otimes L$$

$\Xi; \Theta; \Lambda_1, B_1, B_2 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, B_1 \otimes B_2 \Longrightarrow K \text{ affirms } A$	$\otimes L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \mathbf{1}[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \ \mathbf{1}L$$

$\Xi; \Theta; \Lambda_1 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, \mathbf{1} \Longrightarrow K \text{ affirms } A$	$\mathbf{1}L$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \mathrel{\&} B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \ \mathrel{\&}L_1$$

$\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, B_1 \mathrel{\&} B_2 \Longrightarrow K \text{ affirms } A$	$\mathrel{\&}L_1$ Rule on previous line

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \mathrel{\&} B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \ \mathrel{\&}L_2$$

$\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda_1, B_1 \mathrel{\&} B_2 \Longrightarrow K \text{ affirms } A$	$\mathrel{\&}L_2$ Rule on previous line

Case: The last rule of $\mathcal{D}$ is $\oplus L$, and $\mathcal{D}$ has the form:

$$\frac{\overset{\mathcal{D}_1}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \quad \overset{\mathcal{D}_2}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B_1 \oplus B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}$$

$\Xi; \Theta; \Lambda_1, B_1 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}_1$
$\Xi; \Theta; \Lambda_1, B_2 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}_2$
$\Xi; \Theta; \Lambda_1, B_1 \oplus B_2 \Longrightarrow K \text{ affirms } A$	$\oplus L$ Rule on previous lines

Case: The last rule in $\mathcal{D}$ is $\multimap L$, and $\mathcal{D}$ has the following form:

$$\frac{
\begin{array}{ccc}
\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3
\end{array}
}{}$$

$$\frac{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}] \Longrightarrow B_1[I'''] \quad \Sigma;\Psi \models I' \supseteq I''' \quad \Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_2[\vec{I_2'}], B_2[I'''] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}],\Lambda_2[\vec{I_2'}], B_1 \multimap B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}$$

$\Xi;\Theta;\Lambda_1 \Longrightarrow B_1$	I.H.(2) on $\mathcal{D}_1$
$\Xi;\Theta;\Lambda_2, B_2 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}_3$
$\Xi;\Theta;\Lambda_1,\Lambda_2, B_1 \multimap B_2 \Longrightarrow K \text{ affirms } A$	$\multimap L$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}'}{\Sigma;\Psi;\Theta[\![\vec{I}]\!], B[\![I']\!];\Lambda_1[\vec{I_1'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}}{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}], !B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \;\; !L$$

$\Xi;\Theta, B;\Lambda_1 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}'$
$\Xi;\Theta;\Lambda_1, !B \Longrightarrow K \text{ affirms } A$	$!L$ Rule on previous line

Case: The last rule in $\mathcal{D}$ is $\supset L$ and $\mathcal{D}$ has the form:

$$\frac{
\begin{array}{ccc}
\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3
\end{array}
}{}$$

$$\frac{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\cdot \Longrightarrow B_1[I'''] \quad \Sigma;\Psi \models I' \supseteq I''' \quad \Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}], B_2[I'''] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}], B_1 \supset B_2[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''}$$

$\Xi;\Theta;\cdot \Longrightarrow B_1$	I.H.(2) on $\mathcal{D}_1$
$\Xi;\Theta;\Lambda_1, B_2 \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}_3$
$\Xi;\Theta;\Lambda_1, B_1 \supset B_2 \Longrightarrow K \text{ affirms } A$	$\supset L$ Rule on previous lines

Case:

$$\mathcal{D} = \frac{
\begin{array}{cc}
\mathcal{D}_1 & \mathcal{D}_2
\end{array}
}{}$$

$$\mathcal{D} = \frac{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}], [t/x]B[I'] \Longrightarrow A_1[I''] \quad \Sigma \vdash t{:}s}{\Sigma;\Psi;\Theta[\![\vec{I}]\!];\Lambda_1[\vec{I_1'}], \forall x{:}s.B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \;\; \forall L$$

$\Xi \vdash t{:}s$	$\mathcal{D}_2$ with s that is not interval
$\Xi;\Theta;\Lambda_1, [t/x]B \Longrightarrow K \text{ affirms } A$	I.H.(2) on $\mathcal{D}_1$
$\Xi;\Theta;\Lambda_1, \forall x{:}s.B \Longrightarrow K \text{ affirms } A$	$\forall L$ Rule on previous lines

Case:

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}'\\ \Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow A[I'']\end{array}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda[\vec{I'}] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \text{ affirms}$$

$\Xi; \Theta; \Lambda \Longrightarrow A$ I.H.(1) on $\mathcal{D}'$
$\Xi; \Theta; \Lambda \Longrightarrow K \text{ affirms } A$ affirms Rule on previous line

Case:

$$\mathcal{D} = \cfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I'' & \Sigma; \Psi \models I' \supseteq I''\end{array}}{\Sigma; \Psi; \Theta[\![\vec{I}]\!]; \Lambda_1[\vec{I_1'}], \langle K \rangle B[I'] \Longrightarrow (K \text{ affirms } A) \text{ at } I''} \langle\rangle L$$

$\Xi; \Theta; \Lambda_1, B \Longrightarrow K \text{ affirms } A$ I.H.(2) on $\mathcal{D}_1$
$\Xi; \Theta; \Lambda_1, \langle K \rangle B \Longrightarrow K \text{ affirms } A$ $\langle\rangle L$ Rule on previous line

$\square$